Seminar 6

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Presentation Outline

The Method of Moments – Basic method

Green's functions for electrostatics in 3D and 2D

The Method of Moments – General formulation

Solution by means of weighted residuals

Capacitance problem in 2D for an unbounded region

Poisson's equation and its solution

Poisson's equation is

$$\nabla^2\phi = -\frac{\rho_{\rm v}}{\epsilon_0}.$$

has the solution

$$\phi(\vec{r}) = \int_V \frac{\rho_{\rm v}(\vec{r}\,')}{4\pi\epsilon_0|\vec{r}-\vec{r}\,'|}dV'$$

which is based on the superposition of contributions

$$d\phi(\vec{r}) = \frac{dq'}{4\pi\epsilon_0|\vec{r}-\vec{r}\,'|}$$

with point charges $dq' = \rho_{\rm v}(\vec{r}\,')dV'$ at locations $\vec{r}\,'$.

Integral equations

Known potential $\phi(\vec{r}) = \phi_{\text{spec}}(\vec{r})$ on conductor surfaces S yields integral equation

$$\frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_{\text{s}}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dS' = \phi_{\text{spec}}(\vec{r})$$

to solve for the unknown charge density $\rho_{\text{s}}(\vec{r}\,')$ on the surface of the conductor.

In 2D, the surface integral reduces to a line integral

$$-\frac{1}{2\pi\epsilon_0}\int_S \rho_{\text{l}}(\vec{r}\,')\ln|\vec{r}-\vec{r}\,'|dl' = \phi_{\text{spec}}(\vec{r}).$$

which is based on the superposition of the potential from a line charge

$$d\phi(\vec{r}) = -\frac{\rho_{\text{l}}(\vec{r}\,')dl'}{2\pi\epsilon_0}\ln|\vec{r}-\vec{r}\,'|$$

Presentation Outline

Green's function in 3D

The potential from a point charge in three dimensions satisfies Poisson's equation,

$$-\epsilon_0 \nabla^2 \phi(\vec{r}) = \delta^3(\vec{r} - \vec{r}\,').$$

where $\delta^3(\vec{r} - \vec{r}\,')$ is the 3D Dirac delta function.

In Cartesian coordinates, we have

$$\delta^3(\vec{r} - \vec{r}\,') = \delta(x - x')\delta(y - y')\delta(z - z')$$

where $\delta(\xi - \xi') = 0$ for $\xi \neq \xi'$ such that

$$\int_{\xi_1}^{\xi_2} \delta(\xi - \xi') d\xi = \begin{cases} 1 & \text{if } \xi_1 < \xi' < \xi_2 \\ 0 & \text{otherwise} \end{cases}$$

Green's function in 3D

The Green's function $G(\vec{r}, \vec{r}\,')$ satisfied

$$-\epsilon_0 \nabla_r^2 G(\vec{r}, \vec{r}\,') = \delta^3(\vec{r} - \vec{r}\,')$$

where ∇_r^2 acts on the $\vec{r}$ argument.

By symmetry, we have $G(\vec{r}, \vec{r}\,') = G(R)$ with the distance $R = |\vec{r} - \vec{r}\,'|$ between the source and observation point.

For $R > 0$, we have

$$-\epsilon_0 \frac{1}{R^2} \frac{d}{dR} \left(R^2 \frac{dG}{dR} \right) = 0$$

in spherical coordinates with the origin at the point source.

Green's function in 3D

We have two possible solutions

$$G_1 = a_1 \text{ (rejected since no electric field)}$$
$$G_2 = \frac{a_2}{R}$$

where a_2 is a constant to be determined.

Thus, we have the Green's function

$$G = G_2 = \frac{a_2}{R}$$

Green's function in 3D

Integrate $-\epsilon_0 \nabla_r^2 G(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}')$ over sphere of radius R_0

$$
\begin{aligned}
-\epsilon_0 \int_{R<R_0} \nabla \cdot \nabla G dV &= -\epsilon_0 \oint_{R=R_0} \nabla G \cdot \hat{n} dS \\
&= -\epsilon_0 \left(-\frac{a_2}{R_0^2} \right) \cdot 4\pi R_0^2 \\
&= 4\pi\epsilon_0 a_2 \text{ (left-hand side)} \\
&= 1 \text{ (right-hand side)}
\end{aligned}
$$

which gives $a_2 = 1/(4\pi\epsilon_0)$ and

$$
G(R) = \frac{a_2}{R} = \frac{1}{4\pi\epsilon_0 R} \Rightarrow G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|}
$$

Green's function in 2D

Redo the derivation with cylindrical coordinates for $r > 0$, which gives

$$-\epsilon_0 \frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0$$

with the origin at the point/line source.

We get (after rejecting the constant solution $G_1 = a_1$) that

$$G = G_2 = a_2 \ln r$$

(Note that $G \to \infty$ as $r \to \infty$.)

Green's function in 2D

Integration of $-\epsilon_0 \nabla_r^2 G(\vec{r}, \vec{r}\,') = \delta^2(\vec{r} - \vec{r}\,')$ over a cylinder of radius r_0 and length L gives

$$\begin{aligned}
-\epsilon_0 \int_{r<r_0} \nabla \cdot \nabla G dV &= -\epsilon_0 \oint_{r=r_0} \nabla G \cdot \hat{n} dS \\
&= -\epsilon_0 \frac{a_2}{r_0} \cdot 2\pi r_0 L \text{ (left-hand side)} \\
&= L \text{ (right-hand side)}
\end{aligned}$$

which gives $a_2 = -1/(2\pi\epsilon_0)$ and we get

$$G(r) = -\frac{1}{2\pi\epsilon_0} \ln r \Rightarrow G(\vec{r}, \vec{r}\,') = -\frac{1}{2\pi\epsilon_0} \ln |\vec{r} - \vec{r}\,'|$$

Presentation Outline

General formulation

For a differential equation $L[f] = s$ with a field f related to a source s by means of a differential operator L, we have

$$L_r\left[G(\vec{r},\vec{r}')\right] = \delta^3(\vec{r}-\vec{r}')$$

$$f(\vec{r}) = \int G(\vec{r},\vec{r}')s(\vec{r}')dV'$$

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FEM solution in 3D

Expand the unknown charge distribution $\rho_\mathrm{s}(\vec{r})$ in terms of basis functions $\psi_j(\vec{r})$ and coefficients a_j (to be determined) as

$$\rho_\mathrm{s}(\vec{r}) = \sum_{j=1}^{N} a_j \psi_j(\vec{r})$$

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_\mathrm{s}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dS' &= \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N} a_j \int_S \frac{\psi_j(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dS' \\ &= \sum_{j=1}^{N} a_j \phi_j(\vec{r}) = \phi(\vec{r}) = \phi_\mathrm{spec}(\vec{r})\end{aligned}$$

where the potential ϕ_j due to ψ_j is

$$\phi_j(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\psi_j(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dS'$$

Point matching or collocation

As an example, we subdivide surface into cells with piecewise constant basis functions $\psi_j(\vec{r})$.

At the center $\vec{r}_i$ of each cell i, we require that

$$\phi(\vec{r}_i) = \phi_{\text{spec}}(\vec{r}_i)$$

for $i = 1, 2, \ldots, N$.

This gives a system of linear equations

$$\begin{bmatrix} \phi_1(\vec{r}_1) & \phi_2(\vec{r}_1) & \ldots & \phi_N(\vec{r}_1) \\ \phi_1(\vec{r}_2) & \phi_2(\vec{r}_2) & \ldots & \phi_N(\vec{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\vec{r}_N) & \phi_2(\vec{r}_N) & \ldots & \phi_N(\vec{r}_N) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \phi_{\text{spec}}(\vec{r}_1) \\ \phi_{\text{spec}}(\vec{r}_2) \\ \vdots \\ \phi_{\text{spec}}(\vec{r}_N) \end{bmatrix}$$

Weighted residual

Choose weighting functions $w_i = w_i(\vec{r})$ and require that

$$\int_S w_i(\vec{r}) \left[\phi(\vec{r}) - \phi_{\text{spec}}(\vec{r})\right] dS = 0$$
$$\Rightarrow \quad \langle w_i, \phi \rangle = \langle w_i, \phi_{\text{spec}} \rangle$$

for $i = 1, 2, \ldots, N$.

Galerkin's method: $w_i = \psi_i$
Petrov-Galerkin's method: $w_i \neq \psi_i$
Point matching: $w_i = \delta^2(\vec{r}_i)$

Weighted residual

This gives a system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} \langle w_1, \phi_1 \rangle & \langle w_1, \phi_2 \rangle & \dots & \langle w_1, \phi_N \rangle \\ \langle w_2, \phi_1 \rangle & \langle w_2, \phi_2 \rangle & \dots & \langle w_2, \phi_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_N, \phi_1 \rangle & \langle w_N, \phi_2 \rangle & \dots & \langle w_N, \phi_N \rangle \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \langle w_1, \phi_{\text{spec}} \rangle \\ \langle w_2, \phi_{\text{spec}} \rangle \\ \vdots \\ \langle w_N, \phi_{\text{spec}} \rangle \end{bmatrix}$$

Weighted residual

We have the matrix entries

$$
\begin{aligned}
A_{ij} = \langle w_i, \phi_j \rangle &= \int_S w_i(\vec{r})\phi_j(\vec{r})dS \\
&= \int_S w_i(\vec{r}) \left[\frac{1}{4\pi\epsilon_0} \int_S \frac{\psi_j(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dS' \right] dS \\
&= \frac{1}{4\pi\epsilon_0} \int_S \int_S w_i(\vec{r})\psi_j(\vec{r}\,') \frac{1}{|\vec{r} - \vec{r}\,'|} dS' dS
\end{aligned}
$$

and the vector entries

$$
b_i = \langle w_i, \phi_{\text{spec}} \rangle = \int_S w_i(\vec{r})\phi_{\text{spec}}(\vec{r})dS
$$

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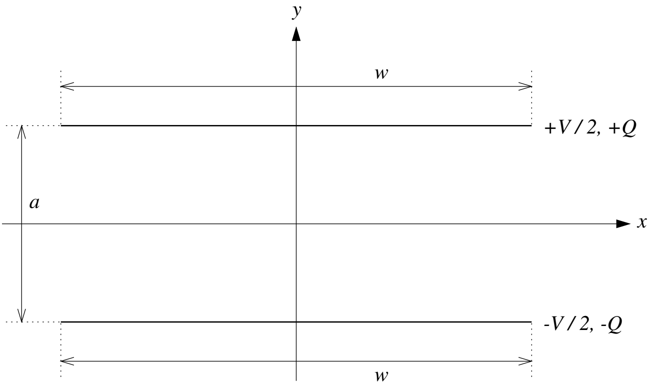
Geometry of parallel plate capacitor



Geometry of parallel plate capacitor

The potential is given by

$$\phi(\vec{r}) = -\frac{1}{2\pi\epsilon_0}\rho_\mathrm{l}(\vec{r}\,') \ln|\vec{r}-\vec{r}\,'|$$
$$\Rightarrow d\phi(\vec{r}) = -\frac{1}{2\pi\epsilon_0}\left[\rho_\mathrm{s}(\vec{r}\,')dl'\right] \ln|\vec{r}-\vec{r}\,'|$$
$$\Rightarrow \phi(\vec{r}) = -\frac{1}{2\pi\epsilon_0}\int_L \rho_\mathrm{s}(\vec{r}\,') \ln|\vec{r}-\vec{r}\,'| dl'$$

and here we get

$$\phi(x,y) = -\frac{1}{2\pi\epsilon_0}\int_{-w/2}^{w/2} \rho_\mathrm{s}\left(x', \frac{a}{2}\right) \ln\sqrt{(x-x')^2 + \left(y-\frac{a}{2}\right)^2}\, dx' - \frac{1}{2\pi\epsilon_0}\int_{-w/2}^{w/2} \rho_\mathrm{s}\left(x', -\frac{a}{2}\right) \ln\sqrt{(x-x')^2 + \left(y+\frac{a}{2}\right)^2}\, dx'$$

Symmetries and discretization

The surface charge density fulfills

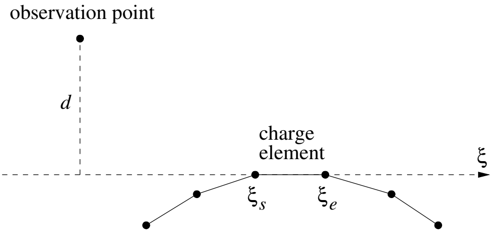
$$\rho_{\mathrm{s}}(-x', a/2) = \rho_{\mathrm{s}}(x', a/2)$$
$$\rho_{\mathrm{s}}(x', -a/2) = -\rho_{\mathrm{s}}(x', a/2)$$

It is enough to discretize only the right half of the upper plate. Use N elements and $\rho_{\mathrm{s}}(x, a/2) = \sum_j \rho_{j+\frac{1}{2}}\, \psi_{j+\frac{1}{2}}(x)$ for $x > 0$.

Discretize each capacitor plate by

- Introduce nodes at $x_j = jh$ with $h = (w/2)/N$ and $j = 0, 1, \ldots, N$
- Define elements on $[x_j, x_{j+1}]$ with $j = 0, 1, \ldots, N-1$
- Piecewise constant basis functions $\psi_{j+\frac{1}{2}}(x)$ (equal to one on element j and zero otherwise)
- Point matching $x_{\mathrm{test},i} = x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$

Potential from one basis functions



The potential is given by

$$I(\xi_\mathrm{s}, \xi_\mathrm{e}, d) = -\frac{1}{2\pi\epsilon_0}\int_{\xi_\mathrm{s}}^{\xi_\mathrm{e}} \ln\sqrt{\xi^2 + d^2}\, dx$$

$$= -\frac{1}{2\pi\epsilon_0}\left[\frac{1}{2}\xi\ln(\xi^2 + d^2) - \xi + d\arctan(\xi/d)\right]_{\xi_\mathrm{s}}^{\xi_\mathrm{e}}$$

for a basis function $\psi_{j+\frac{1}{2}}(\xi')$ that is equal to one on the interval $\xi_\mathrm{s} < \xi' < \xi_\mathrm{e}$ and zero otherwise.

System of linear equations

We have

$$\phi(x_{i+\frac{1}{2}}, y) = \sum_{j=0}^{N-1} A_{ij}\, \rho_{j+\frac{1}{2}}$$

for the testing points $x_{i+\frac{1}{2}}$ for $i = 0, 1, \ldots, N-1$.

The matrix elements are given by

$$\begin{aligned} A_{ij} &= I(x_j - x_{i+\frac{1}{2}}, x_{j+1} - x_{i+\frac{1}{2}}, 0) && \text{upper right quadrant} \\ &+ I(-x_{j+1} - x_{i+\frac{1}{2}}, -x_j - x_{i+\frac{1}{2}}, 0) && \text{upper left quadrant} \\ &- I(x_j - x_{i+\frac{1}{2}}, x_{j+1} - x_{i+\frac{1}{2}}, a) && \text{lower right quadrant} \\ &- I(-x_{j+1} - x_{i+\frac{1}{2}}, -x_j - x_{i+\frac{1}{2}}, a) && \text{lower left quadrant} \end{aligned}$$

and the right-hand side $b_i = \phi_{\text{spec}}(x_{i+\frac{1}{2}}, a/2) = U_0/2$.

Compute the capacitance

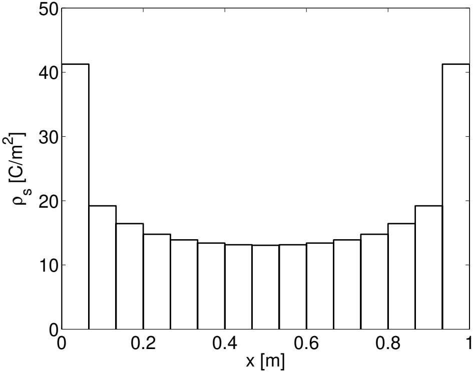
We have the capacitance per unit length as

$$\frac{C}{L} = \frac{Q/L}{U_0} = \frac{h}{U_0} \sum_{j=0}^{N-1} \rho_{j+\frac{1}{2}}$$

Linear convergence in h and $a = w = 1\,\text{m}$ gives (no symm.)

$4N$ [-]	h [m]	C/L [pF/m]
10	0.20000	18.0313850
20	0.10000	18.3729402
30	0.06666	18.4910121
50	0.04000	18.5869926
70	0.02857	18.6285417
100	0.02000	18.6598668
140	0.01428	18.6808279
200	0.01000	18.6965895

Charge distribution – Uniform discretization



Charge distribution – Adaptively refined discretization