Seminar 5

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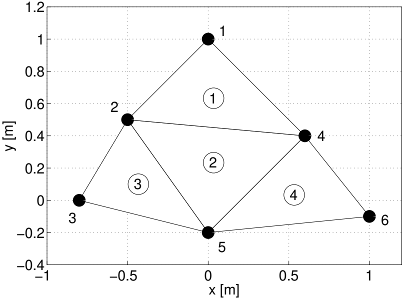
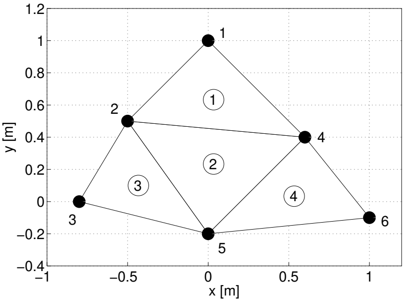
Presentation Outline

Unstructured mesh – nodes



Node	1	2	3	4	5	6
x	0.0	-0.5	-0.8	0.6	0.0	1.0
y	1.0	0.5	0.0	0.4	-0.2	-0.1

Unstructured mesh – elements



Element	1	2	3	4
Node 1	1	2	3	5
Node 2	2	5	5	6
Node 3	4	4	2	4

Unstructured mesh – Matlab

In Matlab, we store

- the coordinates of the nodes
- the nodes of the elements

in matrices as

```
>> no2xy
no2xy =
         0   -0.5000   -0.8000    0.6000         0    1.0000
    1.0000    0.5000         0    0.4000   -0.2000   -0.1000
>> el2no
el2no =
     1     2     3     5
     2     5     5     6
     4     4     2     4
```

Presentation Outline

Unstructured meshes

Assembling procedure

Local element

Element matrices

Waveguide modes (for hollow waveguides)

Sub-divide the integral for the entire domain Ω into the corresponding integrals for the separate elements $\Omega^{(e)}$

$$
\begin{aligned}
A_{ij} = \int_{\Omega} \nabla\varphi_i \cdot \nabla\varphi_j \, dS &= \sum_{e=1}^{N_\mathrm{e}} \int_{\Omega^{(e)}} \nabla\varphi_i \cdot \nabla\varphi_j \, dS \\
&= \sum_{e=1}^{N_\mathrm{e}} \int_{\Omega^{(e)}} \nabla\varphi_i^{(e)} \cdot \nabla\varphi_j^{(e)} \, dS
\end{aligned}
$$

where N_e is the total number of elements. (Note: Some abuse of notation for i and j that are used for both local and global node numbers!)

Here, we have

$$
\begin{aligned}
\varphi_i^{(e)} &= \text{(local) basis function restricted to element } e \\
i &= 1, 2, 3 = \text{local node indices for element } e
\end{aligned}
$$

Assembling procedure

Introduce the element matrix

$$A_{i,j}^{(e)} = \int_{\Omega^{(e)}} \nabla\varphi_i^{(e)} \cdot \nabla\varphi_j^{(e)} \, dS$$

Addition to the global matrix (without abuse of notation)

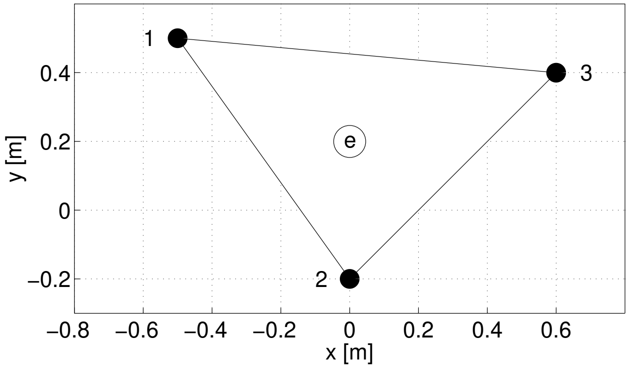
$$A_{n_i,n_j} = A_{n_i,n_j} + A_{i,j}^{(e)}$$

for all elements $e = 1, 2, \ldots, N_\mathrm{e}$ and local node indices $i = 1, 2, 3$ (corresponds to global node indices n_1, n_2, n_3) and $j = 1, 2, 3$ (corresponds to global node indices n_1, n_2, n_3).

Example: Element $e = 2$ with nodes $n_1 = 2$, $n_2 = 5$ and $n_3 = 4$.

Presentation Outline

Local element numbering



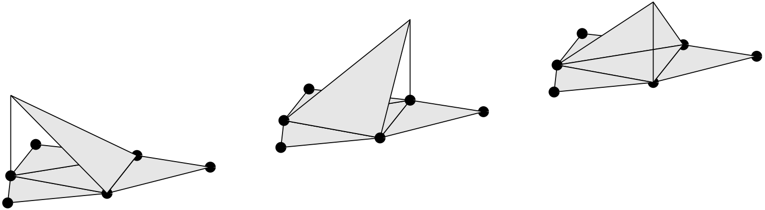
Local basis functions

On element e, we have the basis functions $\varphi_i^{(e)} = \varphi_i^{(e)}(x, y)$ given by

$$\varphi_i^{(e)}\left(x_j^{(e)}, y_j^{(e)}\right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_i^{(e)}(x, y) = a_i^{(e)} + b_i^{(e)} x + c_i^{(e)} y$$

where i and j are local indices that take the values 1, 2 and 3.



Local basis functions

On element e, we have the coefficients for basis functions $\varphi_1^{(e)} = \varphi_i^{(e)}(x, y)$ given by

$$\varphi_1^{(e)}\left(x_1^{(e)}, y_1^{(e)}\right) = a_1^{(e)} + b_1^{(e)} x_1^{(e)} + c_1^{(e)} y_1^{(e)} = 1$$
$$\varphi_1^{(e)}\left(x_2^{(e)}, y_2^{(e)}\right) = a_1^{(e)} + b_1^{(e)} x_2^{(e)} + c_1^{(e)} y_2^{(e)} = 0$$
$$\varphi_1^{(e)}\left(x_3^{(e)}, y_3^{(e)}\right) = a_1^{(e)} + b_1^{(e)} x_3^{(e)} + c_1^{(e)} y_3^{(e)} = 0$$

or

$$\begin{bmatrix} 1 & x_1^{(e)} & y_1^{(e)} \\ 1 & x_2^{(e)} & y_2^{(e)} \\ 1 & x_3^{(e)} & y_3^{(e)} \end{bmatrix} \begin{bmatrix} a_1^{(e)} \\ b_1^{(e)} \\ c_1^{(e)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Local basis functions

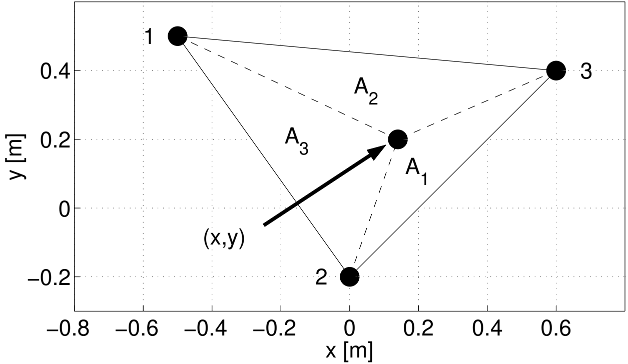
Similar for the two other basis functions, gives the general formula

$$\begin{bmatrix} 1 & x_1^{(e)} & y_1^{(e)} \\ 1 & x_2^{(e)} & y_2^{(e)} \\ 1 & x_3^{(e)} & y_3^{(e)} \end{bmatrix} \begin{bmatrix} a_1^{(e)} & a_2^{(e)} & a_3^{(e)} \\ b_1^{(e)} & b_2^{(e)} & b_3^{(e)} \\ c_1^{(e)} & c_2^{(e)} & c_3^{(e)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which gives

$$\begin{bmatrix} a_1^{(e)} & a_2^{(e)} & a_3^{(e)} \\ b_1^{(e)} & b_2^{(e)} & b_3^{(e)} \\ c_1^{(e)} & c_2^{(e)} & c_3^{(e)} \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(e)} & y_1^{(e)} \\ 1 & x_2^{(e)} & y_2^{(e)} \\ 1 & x_3^{(e)} & y_3^{(e)} \end{bmatrix}^{-1}$$

Local element – Simplex coordinates



Basis functions can be formulated as

$$\varphi_i^{(e)}(x, y) = \frac{A_i^{(e)}}{A_{\text{tot}}^{(e)}}$$

which is identical to the i-th Simplex coordinate.

Local element – Simplex coordinates

Area expressions

$$A_1^{(e)} = \frac{1}{2}\hat{z} \cdot \left(\vec{r}_3^{(e)} - \vec{r}_2^{(e)}\right) \times \left(\vec{r} - \vec{r}_2^{(e)}\right),$$
$$A_2^{(e)} = \frac{1}{2}\hat{z} \cdot \left(\vec{r}_1^{(e)} - \vec{r}_3^{(e)}\right) \times \left(\vec{r} - \vec{r}_3^{(e)}\right),$$
$$A_3^{(e)} = \frac{1}{2}\hat{z} \cdot \left(\vec{r}_2^{(e)} - \vec{r}_1^{(e)}\right) \times \left(\vec{r} - \vec{r}_1^{(e)}\right),$$

with $A_{\text{tot}}^{(e)} = A_1^{(e)} + A_2^{(e)} + A_3^{(e)}$. More compactly

$$A_i^{(e)} = \frac{1}{2}\left(\vec{r} - \vec{r}_{i+1}^{(e)}\right) \cdot \left(\hat{z} \times \vec{s}_i^{(e)}\right)$$
$$A_{\text{tot}}^{(e)} = \frac{1}{2}\hat{z} \cdot \left(\vec{s}_2^{(e)} \times \vec{s}_3^{(e)}\right)$$

where

$$\vec{s}_i^{(e)} = \vec{r}_{i-1}^{(e)} - \vec{r}_{i+1}^{(e)}$$

Gradient of the basis function

The gradient is

$$\nabla \varphi_i^{(e)} = \frac{\hat{z} \times \vec{s}_i^{(e)}}{2A_{\text{tot}}^{(e)}},$$

which is constant on the triangle.

It can also be expressed as

$$\begin{aligned}\nabla \varphi_i^{(e)}(x, y) &= \nabla \left(a_i^{(e)} + b_i^{(e)} x + c_i^{(e)} y \right) \\ &= \hat{x} b_i^{(e)} + \hat{y} c_i^{(e)}\end{aligned}$$

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Stiffness matrix for the Laplacian operator

For the Laplacian, the local stiffness element-matrix is

$$A_{ij}^{(e)} = \int_{\Omega^{(e)}} \nabla\varphi_i^{(e)} \cdot \nabla\varphi_j^{(e)} \, d\Omega = \frac{\vec{s}_i^{(e)} \cdot \vec{s}_j^{(e)}}{4A_{\text{tot}}^{(e)}}.$$

which requires a constant material parameter α.

It can also be expressed as

$$\begin{aligned} A_{ij}^{(e)} &= \int_{\Omega^{(e)}} \left(\hat{x}b_i^{(e)} + \hat{y}c_i^{(e)}\right) \cdot \left(\hat{x}b_j^{(e)} + \hat{y}c_j^{(e)}\right) \, d\Omega \\ &= \left(b_i^{(e)}b_j^{(e)} + c_i^{(e)}c_j^{(e)}\right) A_{\text{tot}}^{(e)} \end{aligned}$$

Mass matrix

For the identity operator, the local mass element-matrix is

$$B_{ij}^{(e)} = \int_{\Omega^{(e)}} \varphi_i^{(e)} \varphi_j^{(e)} \; d\Omega$$

which requires a constant material parameter β.

Here, the following result is useful

$$\int_{\Omega^{(e)}} \left(\varphi_1^{(e)}\right)^{\alpha} \left(\varphi_2^{(e)}\right)^{\beta} \left(\varphi_3^{(e)}\right)^{\gamma} \; d\Omega = 2A_{\text{tot}}^{(e)} \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}$$

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Transverse electric (TE) modes

For a transverse electric (TE) mode, we have computed $H_z(x, y)$ and, then, the corresponding electromagnetic field (that propagates in the $\pm\hat{z}$-direction) is given by

$$\vec{E}(x, y, z) = \Big[+ \frac{j\omega\mu_0}{k_t^2} \hat{z} \times \nabla H_z(x, y) \Big] e^{\mp jk_z z}$$

$$\vec{H}(x, y, z) = \Big[\mp \frac{jk_z}{k_t^2} \nabla H_z(x, y) + \hat{z} H_z(x, y) \Big] e^{\mp jk_z z}$$

Transverse magnetic (TM) modes

For a transverse magnetic (TM) mode, we have computed $E_z(x, y)$ and, then, the corresponding electromagnetic field (that propagates in the $\pm\hat{z}$-direction) is given by

$$\vec{E}(x, y, z) = \Big[\mp \frac{jk_z}{k_t^2} \nabla E_z(x, y) + \hat{z} E_z(x, y) \Big] e^{\mp jk_z z}$$

$$\vec{H}(x, y, z) = \Big[-\frac{j\omega\epsilon_0}{k_t^2} \hat{z} \times \nabla E_z(x, y) \Big] e^{\mp jk_z z}$$

Rectangular waveguide

Consider a rectangular waveguide with a cross section of width a and height b that occupies the region described by $0 \leq x \leq a$ and $0 \leq y \leq b$ for a plane with constant z-coordinate.

We have the transverse wavenumber

$$k_t = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

where m and n are integers.

Longitudinal magnetic field component

Consider a rectangular waveguide that occupies the region described by $0 \leq x \leq a$ and $0 \leq y \leq b$.

The longitudinal magnetic field component of the TE_{mn}-mode is given by

$$H_z(x, y) = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

where $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$ excluding the combination $m = n = 0$.

Longitudinal electric field component

Consider a rectangular waveguide that occupies the region described by $0 \leq x \leq a$ and $0 \leq y \leq b$.

The longitudinal electric field component of the TM_{mn}-mode is given by

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$ for any combinations of m and n.