Seminar 4

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Presentation Outline

General recipe

Solve partial differential equation $L[f] = s$ with differential operator L, sought field f and source s by the procedure:

- Subdivide the solution domain Ω into cells, or *elements*.
- Approximate the solution by an expansion in a finite number of *basis functions*, i.e., $f(\vec{r}) \approx \sum_{i=1}^{n} f_i \varphi_i(\vec{r})$, where f_i are (unknown) coefficients multiplying the basis functions $\varphi_i(\vec{r})$.
- Form the residual $r = L[f] - s$, which we want to make as small as possible.
- Choose *test*, or *weighting*, functions w_i, $i = 1, 2, \ldots, n$ for weighting the residual r. Galerkin's method: $w_i = \varphi_i$
- Set the weighted residuals to zero and solve for the unknowns f_i, i.e. solve $\langle w_i, r \rangle = \int_\Omega w_i \, r \, d\Omega = 0$, $i = 1, 2, \ldots, n$.

Presentation Outline

Model problem

Model problem in 1D given by

$$
\begin{aligned}
-\frac{d}{dx}\left(\alpha\frac{df}{dx}\right)+\beta f &= s && \text{for } a<x<b\\
f &= p && \text{at } x=a\\
\alpha\frac{df}{dx}+\gamma f &= q && \text{at } x=b
\end{aligned}
$$

with

- $f = f(x)$ = sought field
- $\alpha = \alpha(x)$ and $\beta = \beta(x)$ = coefficients that involve material parameters, frequency, etc
- $s = s(x)$ = source differential equation

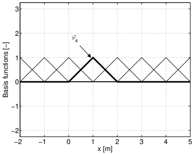
Boundary conditions

- Dirichlet boundary condition $f = p$
  - Specifies the value of f on the boundary of the domain
  - $p = 0 \Rightarrow$ homogeneous Dirichlet boundary condition
  - $p \neq 0 \Rightarrow$ inhomogeneous Dirichlet boundary condition
- Robin boundary condition $\alpha df/dx + \gamma f = q$
  - Specifies a linear combination of the value of f and its derivative on the boundary of the domain
  - Special case with $\gamma = 0$ is referred to as a Neumann boundary condition, which only specifies the derivative of f
  - $q = 0 \Rightarrow$ homogeneous Robin/Neumann boundary condition
  - $q \neq 0 \Rightarrow$ inhomogeneous Robin/Neumann boundary condition

Example – first hand-in assignment

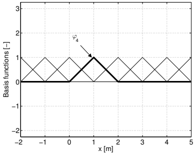
- $f = f(x) = E_z(x)$
- $\alpha = 1$
- $\beta = \beta(x) = \mu_0[j\omega\sigma(x) - \omega^2\epsilon(x)]$
- $s = 0$
- Boundary condition at the left boundary: inhomogeneous Robin boundary condition ($\alpha df/dx + \gamma f = q$)
  - $\gamma = -jk_0$
  - $q = q(x) = -2jk_0E_0 \exp(-jk_0x)$
- Boundary condition at the right boundary: homogeneous Robin boundary condition ($\alpha df/dx + \gamma f = q$)
  - $\gamma = +jk_0$
  - $q = 0$
- (Another problem: metal boundary at the right boundary gives homogeneous Dirichlet boundary condition with $f = p = 0$.)

Follow the recipe



- Subdivide the solution domain Ω into elements:
  - Introduce nodes x_i for $i = 1, 2, \ldots, n$
    (Consider a small example with $n = 8$, see figure above.)
  - Form elements on sub-intervals $[x_i, x_{i+1}]$ for
    $i = 1, 2, \ldots, n - 1$
- Approximate the solution by $f(x) \approx \sum_{j=1}^{n} f_j \varphi_j(x)$, where f_j are (unknown) coefficients multiplying the basis functions $\varphi_j(x)$. (Here, piecewise linears.)

Lowest-order basis functions



Piecewise linear basis functions $\varphi_j(x)$ that fulfill

$$\varphi_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if } x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if } x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Follow the recipe

- Form the residual $r = L[f] - s$, which we want to make as small as possible. Here, we have

$$r = L[f] - s = -\frac{d}{dx}\left(\alpha \frac{df}{dx}\right) + \beta f - s$$

- Choose *test*, or *weighting*, functions w_i, $i = 1, 2, \ldots, n$ for weighting the residual r. (Here, Galerkin's method with $w_i = \varphi_i$.)
- Set the weighted residuals to zero and solve for the unknowns f_i, i.e. solve $\langle w_i, r \rangle = \int_\Omega w_i \, r \, d\Omega = 0$, $i = 1, 2, \ldots, n$.

Integration by parts

$$\langle w_i, r \rangle = \int_\Omega w_i \, r \, d\Omega = \int_a^b w_i \left[-\frac{d}{dx}\left(\alpha \frac{df}{dx} \right) + \beta f - s \right] dx$$
$$= \int_a^b \left[-w_i \frac{d}{dx}\left(\alpha \frac{df}{dx} \right) + \beta w_i f - w_i s \right] dx$$

Integration by parts

$$\int_a^b u(x) v'(x) dx = [u(x)v(x)]_a^b - \int_a^b u'(x) v(x) dx$$
$$u(x) = w_i(x)$$
$$v(x) = -\alpha(x) \frac{df(x)}{dx}$$

Integration by parts

We get

$$\int_a^b \left[-w_i \frac{d}{dx}\left(\alpha \frac{df}{dx}\right)\right] dx = -\left[w_i \left(\alpha \frac{df}{dx}\right)\right]_a^b + \int_a^b \alpha \frac{dw_i}{dx}\frac{df}{dx} dx$$

and

$$\langle w_i, r\rangle = \int_a^b \left[\alpha \frac{dw_i}{dx}\frac{df}{dx} + \beta w_i f - w_i s\right] dx - \left[w_i \left(\alpha \frac{df}{dx}\right)\right]_a^b = 0$$

Recall the boundary conditions

$$f = p \qquad \text{at } x = a$$

$$\alpha \frac{df}{dx} + \gamma f = q \qquad \text{at } x = b$$

Boundary conditions

The Dirichlet boundary condition at $x = a$ implies that all weighting functions are zero there $\Rightarrow w_i(a) = 0$. The solution is already known at $x = a$ so there is no need for weighting.

At $x = b$, we have

$$\alpha \frac{df}{dx} + \gamma f = q \qquad \Rightarrow \qquad \alpha \frac{df}{dx} = q - \gamma f$$

This gives

$$\left[w_i \left(\alpha \frac{df}{dx} \right) \right]_a^b = w_i \left(\alpha \frac{df}{dx} \right) \Bigg|_{x=b} - \underbrace{w_i \left(\alpha \frac{df}{dx} \right) \Bigg|_{x=a}}_{=0} = w_i \left(q - \gamma f \right) \Bigg|_{x=b}$$

Weak form

We get the weak form

$$\langle w_i, r\rangle = \int_a^b \left[\alpha \frac{dw_i}{dx}\frac{df}{dx} + \beta w_i f - w_i s\right] dx - w_i \left(q - \gamma f\right)\Bigg|_{x=b} = 0$$

for all $i = 2, \ldots, n$ given that f is known at $x = a$.

Form equations according to

$$\begin{aligned}
\langle w_2, r\rangle &= 0 \\
\langle w_3, r\rangle &= 0 \\
&\vdots \\
\langle w_n, r\rangle &= 0
\end{aligned}$$

System of linear equations

The small problem with $n = 8$ (with piecewise linears) gives

$$\int_a^b \left[\alpha \frac{dw_2}{dx}\frac{df}{dx} + \beta w_2 f - w_2 s\right] dx = 0$$

$$\int_a^b \left[\alpha \frac{dw_3}{dx}\frac{df}{dx} + \beta w_3 f - w_3 s\right] dx = 0$$

$$\vdots$$

$$\int_a^b \left[\alpha \frac{dw_7}{dx}\frac{df}{dx} + \beta w_7 f - w_7 s\right] dx = 0$$

$$\int_a^b \left[\alpha \frac{dw_8}{dx}\frac{df}{dx} + \beta w_8 f - w_8 s\right] dx - (q - \gamma f)\Big|_{x=b} = 0$$

System of linear equations

The small problem with $n = 8$ gives

$$\int_a^b \left[\alpha \frac{dw_2}{dx}\frac{df}{dx} + \beta w_2 f\right] dx = \int_a^b w_2 s dx$$

$$\int_a^b \left[\alpha \frac{dw_3}{dx}\frac{df}{dx} + \beta w_3 f\right] dx = \int_a^b w_3 s dx$$

$$\vdots$$

$$\int_a^b \left[\alpha \frac{dw_7}{dx}\frac{df}{dx} + \beta w_7 f\right] dx = \int_a^b w_7 s dx$$

$$\int_a^b \left[\alpha \frac{dw_8}{dx}\frac{df}{dx} + \beta w_8 f\right] dx + \gamma f\Big|_{x=b} = \int_a^b w_8 s dx + q\Big|_{x=b}$$

System of linear equations

The solution is expanded as

$$f(x) = \sum_{j=1}^{n} f_j \varphi_j(x)$$

where it is known that $f(a) = p$ or

$$f(a) = \sum_{j=1}^{n} f_j \varphi_j(a) = f_1 = p \quad \Rightarrow \quad f_1 = p$$

For the small problem with $n = 8$, we have also (on the right boundary) that

$$f(b) = \sum_{j=1}^{n} f_j \varphi_j(b) = f_8$$

System of linear equations

The solution is expanded as $f(x) = \sum_{j=1}^{n} f_j \varphi_j(x)$.
Galerkin's method implies $w_i(x) = \varphi_i(x)$, which yields

$$\sum_{j=1}^{8} f_j \int_a^b \left[\alpha \frac{d\varphi_2}{dx} \frac{d\varphi_j}{dx} + \beta \varphi_2 \varphi_j \right] dx = \int_a^b \varphi_2 s dx$$

$$\sum_{j=1}^{8} f_j \int_a^b \left[\alpha \frac{d\varphi_3}{dx} \frac{d\varphi_j}{dx} + \beta \varphi_3 \varphi_j \right] dx = \int_a^b \varphi_3 s dx$$

$$\vdots$$

$$\sum_{j=1}^{8} f_j \int_a^b \left[\alpha \frac{d\varphi_7}{dx} \frac{d\varphi_j}{dx} + \beta \varphi_7 \varphi_j \right] dx = \int_a^b \varphi_7 s dx$$

$$\sum_{j=1}^{8} f_j \int_a^b \left[\alpha \frac{d\varphi_8}{dx} \frac{d\varphi_j}{dx} + \beta \varphi_8 \varphi_j \right] dx + \gamma(b) f_8 = \int_a^b \varphi_8 s dx + q(b)$$

On matrix form $\mathbf{Ax} = \mathbf{b}$, we have

$$\underbrace{\begin{bmatrix} \xi_{22} & \xi_{23} & 0 & 0 & 0 & 0 & 0 \\ \xi_{32} & \xi_{33} & \xi_{34} & 0 & 0 & 0 & 0 \\ 0 & \xi_{43} & \xi_{44} & \xi_{45} & 0 & 0 & 0 \\ 0 & 0 & \xi_{54} & \xi_{55} & \xi_{56} & 0 & 0 \\ 0 & 0 & 0 & \xi_{65} & \xi_{66} & \xi_{67} & 0 \\ 0 & 0 & 0 & 0 & \xi_{76} & \xi_{77} & \xi_{78} \\ 0 & 0 & 0 & 0 & 0 & \xi_{87} & \xi_{88} + \gamma \end{bmatrix}}_{=\mathbf{A}} \underbrace{\begin{bmatrix} f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}}_{=\mathbf{x}} = \underbrace{\begin{bmatrix} \zeta_2 - \xi_{21} p \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \\ \zeta_6 \\ \zeta_7 \\ \zeta_8 + q \end{bmatrix}}_{=\mathbf{b}}$$

where $p = p(a)$, $\gamma = \gamma(b)$, $q = q(b)$ and

$$\xi_{ij} = \int_a^b \left[\alpha \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + \beta \varphi_i \varphi_j \right] dx$$

$$\zeta_i = \int_a^b \varphi_i s dx$$

Presentation Outline

Finite element method – General recipe

1D Finite Element Analysis

2D Finite Element Analysis

Model problem

Model problem in 2D given by

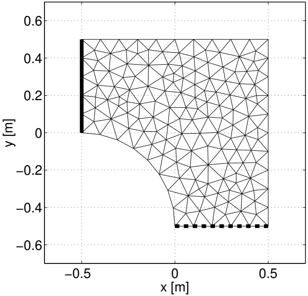
$$\begin{aligned} -\nabla \cdot (\alpha \nabla f) + \beta f &= s && \text{in } \Omega \\ f &= p && \text{at } \partial\Omega_\mathrm{D} \\ \hat{n} \cdot (\alpha \nabla f) + \gamma f &= q && \text{at } \partial\Omega_\mathrm{R} \end{aligned}$$

with

- $f = f(x, y)$ = sought field
- $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ = coefficients that involve material parameters, frequency, etc
- $s = s(x, y)$ = source differential equation

Triangular discretization – an example

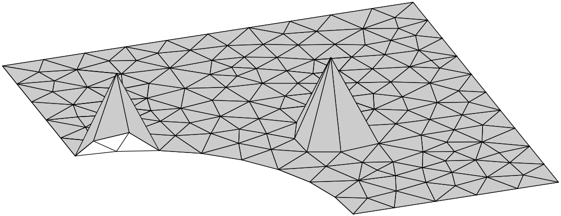
Sub-divide the domain Ω into triangles



Dirichlet boundary conditions $\partial\Omega_{\mathrm{D}}$ on the thick solid and dashed part of the boundary.
Robin boundary condition $\partial\Omega_{\mathrm{R}}$ on the other parts of $\partial\Omega$.

Basis functions (and weighting functions)

Two different piecewise linear basis functions $\varphi_j(\vec{r})$



that fulfill

$$\varphi_j(\vec{r}_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Weak form

Multiply the partial differential equation (PDE) with weighting function w_i and integrate over Ω

$$\int_\Omega w_i \left[-\nabla \cdot (\alpha \nabla f) + \beta f\right] dS = \int_\Omega w_i s \, dS$$

Integrate by parts by means of product rule and Gauss' theorem

$$\nabla \cdot [w_i(\alpha \nabla f)] = \alpha \nabla w_i \cdot \nabla f + w_i \nabla \cdot (\alpha \nabla f)$$

$$\int_\Omega \nabla \cdot \vec{F} dS = \oint_{\partial\Omega} \hat{n} \cdot \vec{F} \, dl$$

Weak form

We get

$$\int_\Omega w_i \left[-\nabla \cdot (\alpha \nabla f)\right] dS = \int_\Omega \left(\alpha \nabla w_i \cdot \nabla f - \nabla \cdot [w_i(\alpha \nabla f)]\right) dS$$
$$= \int_\Omega \alpha \nabla w_i \cdot \nabla f dS - \oint_{\partial\Omega} \hat{n} \cdot [w_i(\alpha \nabla f)] dl$$

Recall the boundary conditions

$$f = p \qquad \text{at } \partial\Omega_\mathrm{D}$$
$$\hat{n} \cdot (\alpha \nabla f) + \gamma f = q \qquad \text{at } \partial\Omega_\mathrm{R}$$

where $\partial\Omega = \partial\Omega_\mathrm{D} + \partial\Omega_\mathrm{R}$.

Weak form

The boundary term yiels

$$\oint_{\partial\Omega} \hat{n} \cdot [w_i(\alpha\nabla f)]dl = \int_{\partial\Omega_\mathrm{D}} \hat{n} \cdot [\underbrace{w_i}_{=0}(\alpha\nabla f)]dl + \int_{\partial\Omega_\mathrm{R}} \hat{n} \cdot [w_i(\alpha\nabla f)]dl$$
$$= \int_{\partial\Omega_\mathrm{R}} w_i \, [\hat{n} \cdot (\alpha\nabla f)]dl = \int_{\partial\Omega_\mathrm{R}} w_i \, [q - \gamma f]dl$$

We get the weak form

$$\int_\Omega [\alpha\nabla w_i \cdot \nabla f + \beta w_i f]dS + \int_{\partial\Omega_\mathrm{R}} w_i \gamma f \, dl = \int_\Omega w_i s \, dS + \int_{\partial\Omega_\mathrm{R}} w_i q \, dl$$

System of linear equations

Expansion $f(\vec{r}) = \sum_{j=1}^{n} f_j \varphi_j(\vec{r})$ and Galerkin's method $w_i(\vec{r}) = \varphi_i(\vec{r})$ gives

$$\sum_{j=1}^{n} f_j \left[\int_{\Omega} [\alpha \nabla \varphi_i \cdot \nabla \varphi_j + \beta \varphi_i \varphi_j] dS + \int_{\partial \Omega_{\mathrm{R}}} \gamma \varphi_i \varphi_j \, dl \right] = \int_{\Omega} \varphi_i s \, dS + \int_{\partial \Omega_{\mathrm{R}}} \varphi_i q \, dl$$

for all $i = 1, 2, \ldots, n - n_{\partial \Omega_{\mathrm{D}}}$ with $n_{\partial \Omega_{\mathrm{D}}}$ is the number of nodes where we have a Dirichlet boundary condition $f = p$.

System of linear equations

We get

$$\sum_{j=1}^{n} A_{ij} f_j = b_i \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

with

$$A_{ij} = \int_{\Omega} [\alpha \nabla \varphi_i \cdot \nabla \varphi_j + \beta \varphi_i \varphi_j] dS + \int_{\partial\Omega_{\mathrm{R}}} \gamma \varphi_i \varphi_j \, dl$$

$$b_i = \int_{\Omega} \varphi_i s \, dS + \int_{\partial\Omega_{\mathrm{R}}} \varphi_i q \, dl$$

Note that **A** has more columns than rows if we have a Dirichlet boundary condition!

System of linear equations

Partition the matrix $\mathbf{A}$ and the vector $\mathbf{x}$

$$\left(\begin{array}{c|c} \mathbf{A_n} & \mathbf{A_e} \end{array}\right) \begin{bmatrix} \mathbf{x_n} \\ \mathbf{x_e} \end{bmatrix} = \mathbf{A_n x_n} + \mathbf{A_e x_e} = \mathbf{b}$$

- $\mathbf{x_n}$ contains to the unknowns associated with the interior nodes or nodes on the Robin boundary
- $\mathbf{x_e}$ contains the known coefficients on the Dirichlet boundary

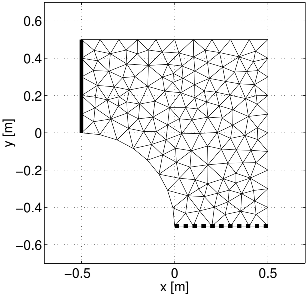
Here, $\mathbf{A_n}$ is a square matrix (invertable) and $\mathbf{A_e}$ is a rectangular matrix.

This gives

$$\mathbf{A_n x_n} = \mathbf{b} - \mathbf{A_e x_e} \quad \Rightarrow \quad \mathbf{x_n} = \mathbf{A_n^{-1}}(\mathbf{b} - \mathbf{A_e x_e})$$

Example

Sub-divide the domain Ω into triangles



Dirichlet boundary conditions on (i) $\partial\Omega_{\mathrm{D},1}$ – the thick solid part of the boundary and (ii) $\partial\Omega_{\mathrm{D},2}$ – the thick solid part of the boundary

Robin boundary condition $\partial\Omega_{\mathrm{R}}$ on the other parts of $\partial\Omega$.

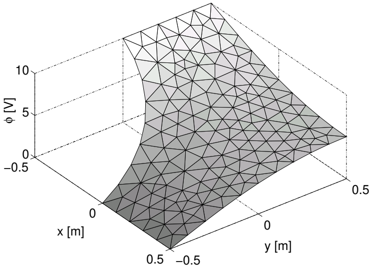
Example

Compute the resistance between the left and bottom edges of the conducting plate

- $f(x, y)$ = the electrostatic potential $\phi = \phi(x, y)$
- $\alpha(x, y)$ = the conductivity σ (constant here)
- $\beta = 0$
- $s = 0$
- $f = 10$ V on $\partial\Omega_{\mathrm{D},1}$ (thick solid line)
- $f = 0$ V on $\partial\Omega_{\mathrm{D},2}$ (thick dashed line)
- $\hat{n} \cdot \nabla f = 0$ on $\partial\Omega_{\mathrm{R}}$
  (Insulating material: a homogeneous Neumann boundary condition – no flux of charge across the boundary)

Example

Electrostatic potential $\phi = \phi(x, y)$ on the domain Ω



For a plate of height h, we have

$$P = \int_V \vec{J} \cdot \vec{E} \, dV = \int_V \sigma |\nabla \phi|^2 \, dV = h\mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{x} = h\mathbf{x}^\mathbf{T}\mathbf{b} \Rightarrow R = \frac{U^2}{P}$$