Finite element method

Hand-in assignment # 4 – SSY200

1 Problem description

A ridge waveguide has a cross section designed to allow a single mode of propagation over larger bandwidths than a rectangular waveguide. A typical cross section of a ridge waveguide is shown in Fig. 1.

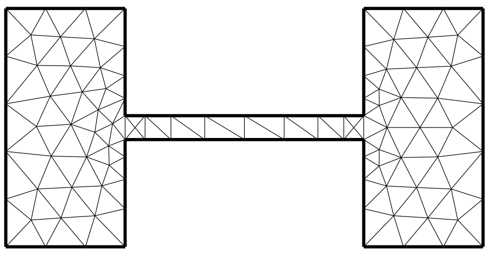

Figure 1: The cross section of a ridge waveguide discretized by triangular finite elements.

To compute the cut-off frequencies, we solve the eigenvalue problem

$$-\nabla^2 H_z = k_t^2 H_z \quad \text{in } S \tag{1}$$
$$\hat{n} \cdot \nabla H_z = 0 \quad \text{on } L \tag{2}$$

for the transverse electric (TE) modes. For the transverse magnetic (TM) modes, we solve

$$-\nabla^2 E_z = k_t^2 E_z \quad \text{in } S \tag{3}$$
$$E_z = 0 \quad \text{on } L \tag{4}$$

where S is the interior of the waveguide and L its boundary. The transverse wavenumber is denoted k_t and the longitudinal wavenumber k_z, i.e. $k^2 = (\omega/c_0)^2 = k_t^2 + k_z^2$. For more information on the theory of waveguides can be found in the literature, cf. [1].

To apply FEM to the eigenvalue problems given above we derive the weak form. The weak form for the two eigenvalue problems are shown below.

$$\int_S \nabla w_i \cdot \nabla H_z \, dS = k_t^2 \int_S w_i H_z \, dS$$
$$\int_S \nabla w_i \cdot \nabla E_z \, dS = k_t^2 \int_S w_i E_z \, dS$$

The z-component of the electric and magnetic fields are expanded in and tested by nodal basis functions ϕ_i. The testing must be done in accordance with the boundary conditions. Observe that for the TE-modes we have a Neumann boundary condition while for the TM-modes we have a Dirichlet boundary condition.

To assemble the matrices of the right and left hand sides we sum contributions from each triangle, i.e. we need to compute the following integrals

$$\begin{aligned} A_{ij}^e &= \int_{S^e} \nabla\phi_i \cdot \nabla\phi_j \, dS \\ B_{ij}^e &= \int_{S^e} \phi_i \phi_j \, dS \end{aligned}$$

2 Assignments

Derive the weak formulations and calculate analytical expressions for A_{ij}^e and B_{ij}^e. The derivations must be included in the report. You might find the following formula useful

$$\int_{S^e} (\phi_1^e)^\alpha (\phi_2^e)^\beta (\phi_3^e)^\gamma \, dS = 2S^e \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} \tag{5}$$

where S^e is the area of element e and the constants α, β and γ take the integer values $0, 1, 2, \ldots$, where you may choose useful combinations of these integers yourself.

2.1 Numerical implementation

The `tar`-file contains the following files:

- `Main.m` : Reads the grid, assembles the matrices and solves the eigenvalue problem.
- `CmpElMtx.m` : Implement your computation of the element matrices here.
- `ReadGrid.m` : Implement your reading of the meshes here.
- `VisualizeMode.m` : Implement your visualization of the eigenmode in this function. This function should visualize two fields: (i) $\psi(x,y)$ by means of colors; and (ii) $\nabla\psi(x,y)$ and/or $\hat{z} \times \nabla\psi(x,y)$ by vectors. Here, $\psi(x,y) = E_z(x,y)$ for the TM-modes and $\psi(x,y) = H_z(x,y)$ for the TE-modes. (These expressions are related to the actual electromagnetic fields for TM- and TE-modes, which are listed in the Appendix A.)

2.2 Meshes

The directory contains meshes stored in text files. The meshes discretize both a rectangular waveguide and a ridge waveguide. For each geometry, there is three discretizations that can be used for convergence studies.

- Meshes for rectangular waveguide of width $l_x = 2$ cm and height $l_y = 1$ cm.
  - `grid_rectangular_res1.txt` – coarse mesh
  - `grid_rectangular_res2.txt` – once hierarchically refined mesh

- `grid_rectangular_res3.txt` – twice hierarchically refined mesh

- Mesh for a ridge waveguide of outer dimensions 2 cm and 1 cm. The spacing between teeth is 0.1 cm and their width is 1 cm. The mesh is shown in Fig. 1.
  - `grid_ridge_res1.txt` – coarse mesh
  - `grid_ridge_res2.txt` – once hierarchically refined mesh
  - `grid_ridge_res3.txt` – twice hierarchically refined mesh

2.3 Numerical tests

Compute the 20 lowest k_t and their corresponding cut-off frequencies ω^{co} for the rectangular waveguide and compare to the analytical expression

$$k_t = \frac{\omega^{\text{co}}}{c_0} = \sqrt{\left(\frac{\pi n_x}{l_x}\right)^2 + \left(\frac{\pi n_y}{l_y}\right)^2} \tag{6}$$

where $n_x = 0, 1, \ldots$ and $n_y = 0, 1, \ldots$ excluding $n_x = n_y = 0$.

- Perform a convergence test for the lowest eigenmode. Does this cut-off frequency converge to the analytical value? What's the order of convergence?
- Visualize the five lowest eigenmodes. Do the eigenmodes compare well with their analytical counterparts? (The analytical results are shown in Appendix B.)

2.3.1 Optional problems

These problems give credit points if they are correctly solved.

5 credit points
Compute the 20 lowest k_t and their corresponding cut-off frequencies f^{co} for the ridge waveguide. Compare the ratio $f_2^{\text{co}}/f_1^{\text{co}}$ between the two lowest cut-off frequencies ($f_2^{\text{co}} > f_1^{\text{co}}$) for the ridge and rectangular waveguides. Is the bandwidth larger for the ridge waveguide?

5 credit points
Perform a convergence test for the lowest eigenmode. What's the extrapolated cut-off frequency? What's the order of convergence? Do you achieve optimal order of convergence? If not, why is the order of convergence reduced?

5 credit points
Visualize both the longitudinal and transverse field components for the five lowest eigenmodes. Comment on how your visualizations relate to the cut-off frequencies. How does the eigenmodes compare with the rectangular waveguide modes?

3 Report

Compare and explain your findings in the report. It is important that you try to provide mathematical arguments to support your conclusions. Do not forget to

- describe your numerical schemes (and suitable derivations) and their implementation by `MATLAB`-program,
- results for the numerical investigations above.

References

[1] D. K. Cheng, *Field and Wave Electromagnetics*. Reading, MA: Addison-Wesley, 2 ed., 1989.

A Electromagnetic fields for waveguide modes

For the m-th transverse electric (TE) mode, we have computed $H_z(x, y)$ and, then, the corresponding electromagnetic field (that propagates in the $\pm\hat{z}$-direction) is given by

$$\vec{E}(x,y,z) = H_0^{\pm}\Big[+\frac{j\omega\mu_0}{k_t^2}\hat{z}\times\nabla H_z(x,y)\Big]e^{\mp jk_z z} \tag{7}$$

$$\vec{H}(x,y,z) = H_0^{\pm}\Big[\mp\frac{jk_z}{k_t^2}\nabla H_z(x,y)+\hat{z}H_z(x,y)\Big]e^{\mp jk_z z} \tag{8}$$

where the amplitude $H_0^{\pm}$ is associated with the magnetic field and it has the unit A/m.

For the m-th transverse magnetic (TM) mode, we have computed $E_z(x, y)$ and, then, the corresponding electromagnetic field (that propagates in the $\pm\hat{z}$-direction) is given by

$$\vec{E}(x,y,z) = E_0^{\pm}\Big[\mp\frac{jk_z}{k_t^2}\nabla E_z(x,y)+\hat{z}E_z(x,y)\Big]e^{\mp jk_z z} \tag{9}$$

$$\vec{H}(x,y,z) = E_0^{\pm}\Big[-\frac{j\omega\epsilon_0}{k_t^2}\hat{z}\times\nabla E_z(x,y)\Big]e^{\mp jk_z z} \tag{10}$$

where the amplitude $E_0^{\pm}$ is associated with the electric field and it has the unit V/m.

B Longitudinal field for rectangular waveguide

Consider a rectangular waveguide with a cross section of width l_x and height l_y that occupies the region described by $0 \le x \le l_x$ and $0 \le y \le l_y$ for a plane with constant z-coordinate.

The longitudinal magnetic field component of the TE_{mn}-mode is given by

$$H_z(x,y) = H_0\cos\left(\frac{n_x\pi}{l_x}x\right)\cos\left(\frac{n_y\pi}{l_y}y\right). \tag{11}$$

where $n_x = 0, 1, 2, \ldots$ and $n_y = 0, 1, 2, \ldots$ excluding the combination $n_x = n_y = 0$.

The longitudinal electric field component of the TM_{mn}-mode is given by

$$E_z(x, y) = E_0 \sin\left(\frac{n_x \pi}{l_x} x\right) \sin\left(\frac{n_y \pi}{l_y} y\right). \qquad (12)$$

where $n_x = 1, 2, 3, \ldots$ and $n_y = 1, 2, 3, \ldots$ for any combinations of n_x and n_y.