Seminar 3
Finite-difference time-domain scheme

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Presentation Outline

FDTD scheme in 3D

Courant condition

Additional functionality

Yee cell

Discretize Maxwell's equations with centered finite differences.

Ampère's law

$$\epsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} \Rightarrow \begin{cases} \epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \epsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{cases}$$

Faraday's law

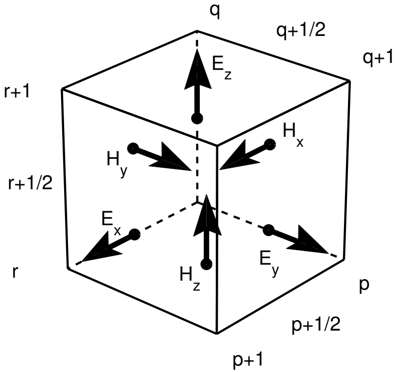
$$\mu \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{E} \Rightarrow \begin{cases} \mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \\ \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \\ \mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \end{cases}$$

Yee cell

The staggering to three dimensions with a special arrangement of all the components of $\vec{E}$ and $\vec{H}$:

- Electric field components are computed at "integer" time-steps
- Magnetic field at "half-integer" time-steps
- Space is divided into bricks with sides Δx, Δy, and Δz (usually one uses cubes with $\Delta x = \Delta y = \Delta z = h$)
- The different field components are placed in the grid according to the Yee cell
- Use the notation $f|^n_{p,q,r} \equiv f(p\Delta x, q\Delta y, r\Delta z, n\Delta t)$ where p, q, r and n are integers

Yee cell



Ampère's law: x-component

Continuum form

$$\epsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

is discretized as

$$\epsilon \frac{E_x|^{n+1}_{p+\frac{1}{2},q,r} - E_x|^{n}_{p+\frac{1}{2},q,r}}{\Delta t}$$

$$= \frac{H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q-\frac{1}{2},r}}{\Delta y} - \frac{H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r-\frac{1}{2}}}{\Delta z}$$

Centered at $p + \frac{1}{2}$, q and r in space and $n + \frac{1}{2}$ in time.

Ampère's law: y- and z-component

Similar for the two other components

$$\epsilon \frac{E_y|^{n+1}_{p,q+\frac{1}{2},r} - E_y|^{n}_{p,q+\frac{1}{2},r}}{\Delta t} = \frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r-\frac{1}{2}}}{\Delta z} - \frac{H_z|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z|^{n+\frac{1}{2}}_{p-\frac{1}{2},q+\frac{1}{2},r}}{\Delta x}$$

$$\epsilon \frac{E_z|^{n+1}_{p,q,r+\frac{1}{2}} - E_z|^{n}_{p,q,r+\frac{1}{2}}}{\Delta t} = \frac{H_y|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y|^{n+\frac{1}{2}}_{p-\frac{1}{2},q,r+\frac{1}{2}}}{\Delta x} - \frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n+\frac{1}{2}}_{p,q-\frac{1}{2},r+\frac{1}{2}}}{\Delta y}$$

Faraday's law: x-component

Continuum form

$$\mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}$$

is discretized as

$$\mu \frac{H_x|^{n+\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}} - H_x|^{n-\frac{1}{2}}_{p,q+\frac{1}{2},r+\frac{1}{2}}}{\Delta t}$$
$$= \frac{E_y|^{n}_{p,q+\frac{1}{2},r+1} - E_y|^{n}_{p,q+\frac{1}{2},r}}{\Delta z} - \frac{E_z|^{n}_{p,q+1,r+\frac{1}{2}} - E_z|^{n}_{p,q,r+\frac{1}{2}}}{\Delta y}$$

Centered at p, $q + \frac{1}{2}$ and $r + \frac{1}{2}$ in space and n in time.

Faraday's law: y- and z-component

Similar for the two other components

$$\mu \frac{H_y\big|^{n+\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}} - H_y\big|^{n-\frac{1}{2}}_{p+\frac{1}{2},q,r+\frac{1}{2}}}{\Delta t}$$

$$= \frac{E_z\big|^{n}_{p+1,q,r+\frac{1}{2}} - E_z\big|^{n}_{p,q,r+\frac{1}{2}}}{\Delta x} - \frac{E_x\big|^{n}_{p+\frac{1}{2},q,r+1} - E_x\big|^{n}_{p+\frac{1}{2},q,r}}{\Delta z}$$

$$\mu \frac{H_z\big|^{n+\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r} - H_z\big|^{n-\frac{1}{2}}_{p+\frac{1}{2},q+\frac{1}{2},r}}{\Delta t}$$

$$= \frac{E_x\big|^{n}_{p+\frac{1}{2},q+1,r} - E_x\big|^{n}_{p+\frac{1}{2},q,r}}{\Delta y} - \frac{E_y\big|^{n}_{p+1,q+\frac{1}{2},r} - E_y\big|^{n}_{p,q+\frac{1}{2},r}}{\Delta x}$$

Presentation Outline

Courant condition

The time-step Δt must fulfil the Courant condition

$$\Delta t \leq \frac{1}{c\sqrt{1/(\Delta x)^2 + 1/(\Delta y)^2 + 1/(\Delta z)^2}}$$

For a cubic grid with $\Delta x = \Delta y = \Delta z = h$, the stability condition simplifies to

$$\Delta t \leq \frac{h}{c\sqrt{3}}.$$

Presentation Outline

FDTD scheme in 3D

Courant condition

Additional functionality

Additional functionality

- Perfectly Matched Layer (PML) absorbs outward propagating waves, which is used for free-space problems. (Allows for grid truncation by a perfect electric conductor.)
- Incident wave created by a Huygen's surface that divides the computational domain into two sub-domains:
  - Total-field region inside the Huygen's surface
  - Scattered-field region outside the Huygen's surface
- Near-to-far-field transformation computes the radiated far-field given the near-fields