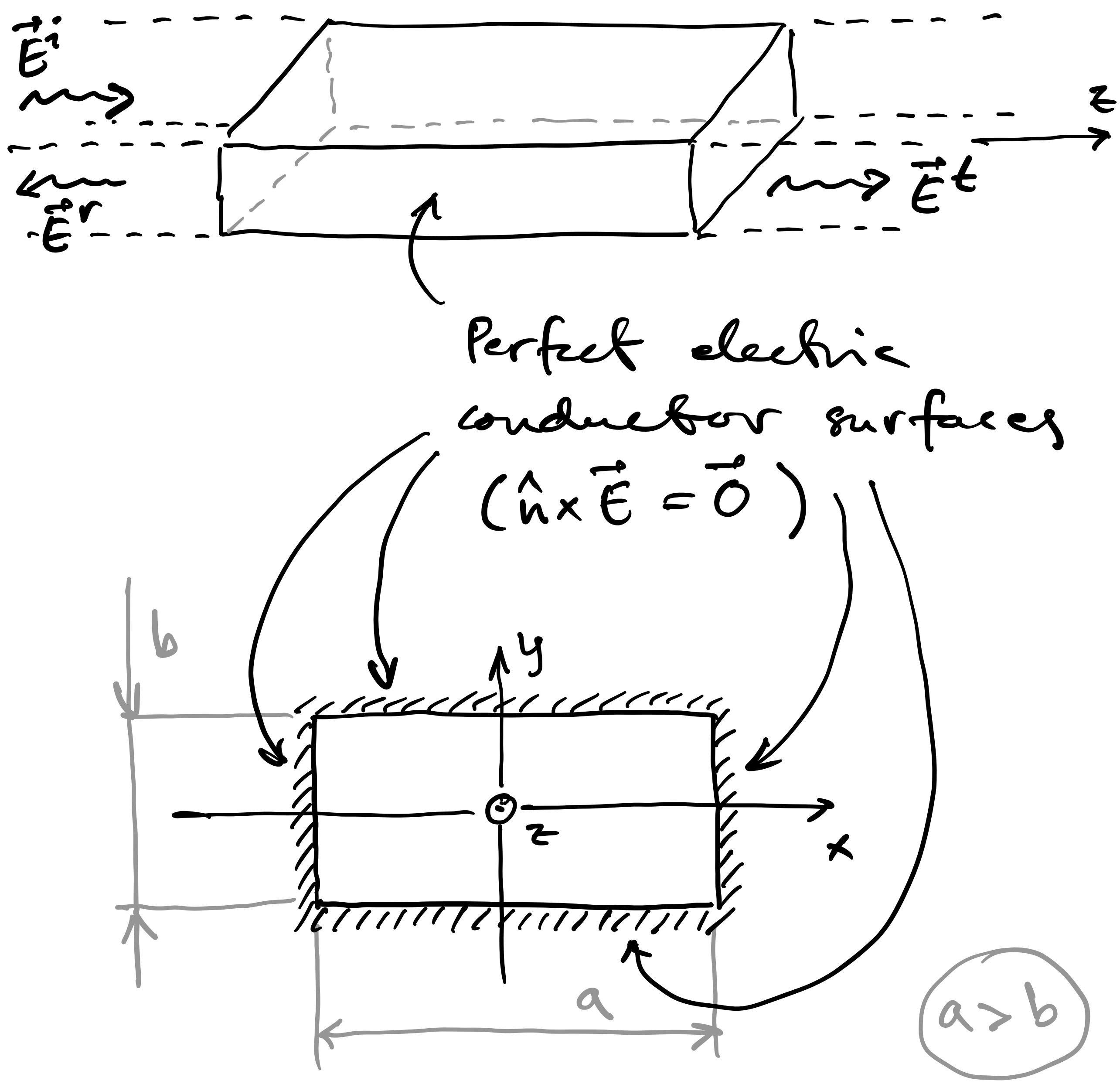
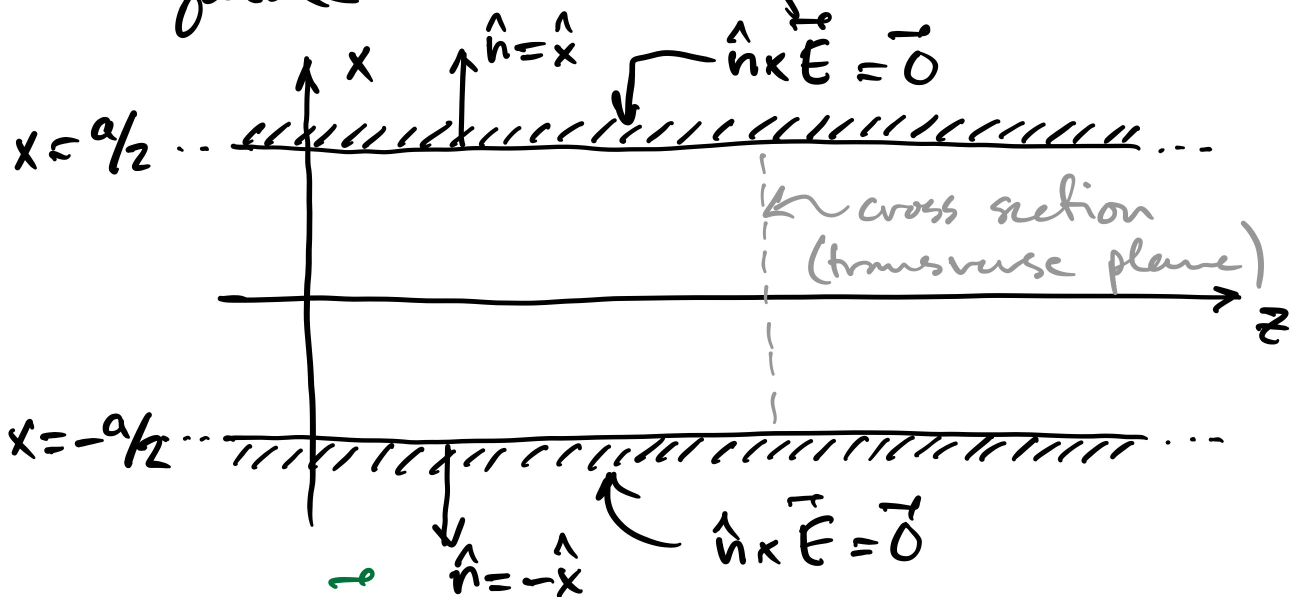
Hand-in assignment #2

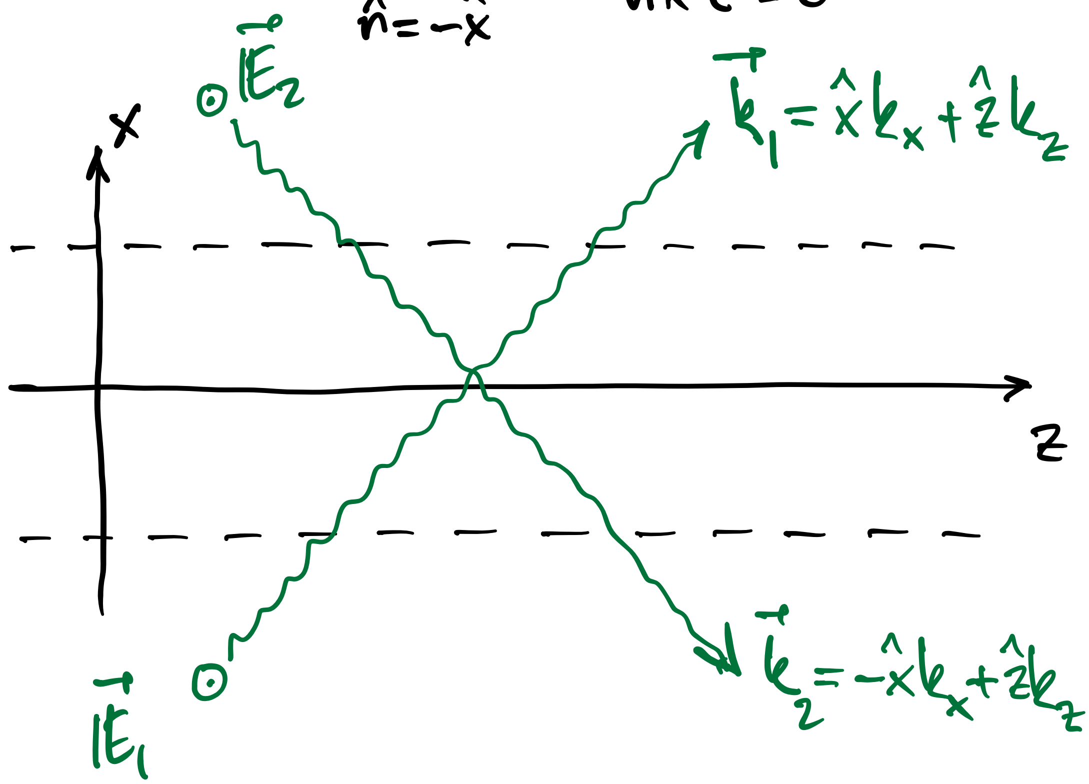


The waveguide allows for propagation along the z-axis 'in' the region

$$-a/2 < x < a/2$$
$$-b/2 < y < b/2$$

Waveguide viewed from above





Superposition of

$$\begin{cases} \vec{E}_1 = \hat{y} E_0 \, e^{-j\vec{k}_1 \cdot \vec{r}} \\ \vec{E}_2 = \hat{y} E_0 \, e^{-j\vec{k}_2 \cdot \vec{r}} \end{cases}$$

yields

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \hat{y} E_0 \left(e^{-j\vec{k}_1 \cdot \vec{r}} + e^{-j\vec{k}_2 \cdot \vec{r}} \right)$$

$$= \left\{ \begin{array}{l} \text{Obs.} \quad \vec{r} = \hat{x}x + \hat{y}y + \hat{z}z \\ \vec{k}_1 \cdot \vec{r} = k_x x + k_z z \\ \vec{k}_2 \cdot \vec{r} = -k_x x + k_z z \end{array} \right\}$$

$$= \hat{y} E_0 \left(e^{-jk_x x} + e^{+jk_x x} \right) e^{-jk_z z}$$

The boundary conditions yield

$$\hat{n} \times \vec{E} \Big|_{x = \pm a/2} = (\pm \hat{x}) \times \vec{E} \Big|_{x = \pm a/2}$$

$$= \pm \hat{z} E_0 \underbrace{\left(e^{-jk_x x} + e^{+jk_x x} \right)}_{} \Big|_{x = \pm a/2} e^{-jk_z z} = \vec{0}$$

$$= 0$$

$$\Rightarrow \quad e^{\mp j k_x a/2} + e^{\pm j k_x a/2} = 0$$

$$\Rightarrow \quad \cos(k_x a/2) \mp \cancel{j \sin(k_x a/2)}$$
$$+ \cos(k_x a/2) \pm \cancel{j \sin(k_x a/2)} = 0$$

$$\Rightarrow \quad 2\cos(\underbrace{k_x a/2}_{= \frac{\pi}{2} + \pi n}) = 0$$

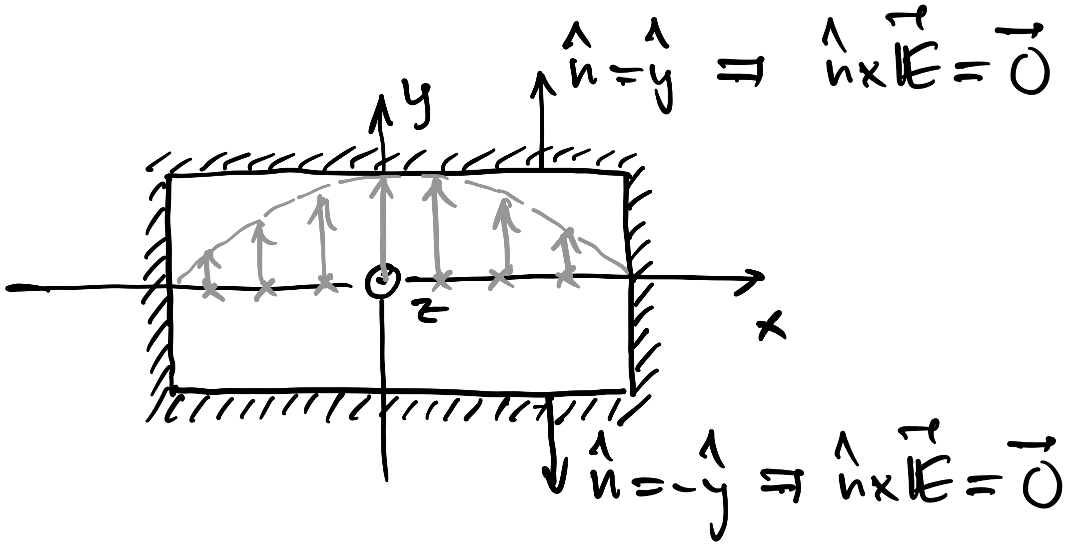
$$\Rightarrow \quad k_x = \frac{\pi}{a} + \frac{2\pi n}{a} \qquad n = 0, 1, 2, \ldots$$

We look for the fundamental waveguide mode and it corresponds to $n = 0$, which gives

$$\vec{E} = \hat{y}\, E_0 \left(e^{-j\pi x/a} + e^{+j\pi x/a} \right) e^{-jk_z z}$$
$$= \hat{y}\, 2E_0 \cos\left(\frac{\pi x}{a}\right) e^{-jk_z z}$$

This is referred to as the

TE_{10}-mode, where TE abbreviates transverse electric.



The magnetic field can be calculated from Faraday's law

$$-j\omega\mu_0 \vec{H} = \nabla \times \vec{E} = -\hat{x}\frac{\partial E_y}{\partial z} + \hat{z}\frac{\partial E_y}{\partial x}$$

$$= 2E_0 \left[-\hat{x}(-jk_z)\cos\left(\frac{\pi x}{a}\right) + \hat{z}\left(\frac{\pi}{a}\right)\left(-\sin\left(\frac{\pi x}{a}\right)\right) \right] e^{-jk_z z}$$

$$\vec{H} = j\frac{2E_0}{\omega\mu_0}\left[\hat{x}\, jk_z \cos\left(\frac{\pi x}{a}\right) - \hat{z}\frac{\pi}{a}\sin\left(\frac{\pi x}{a}\right) \right] e^{-jk_z z}$$

In time-domain

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\vec{\mathbb{E}}(\vec{r})e^{j\omega t}\right\}$$

$$\vec{H}(\vec{r},t) = \mathrm{Re}\left\{\vec{\mathbb{H}}(\vec{r})e^{j\omega t}\right\}$$

Dispersion relation can be calculated from

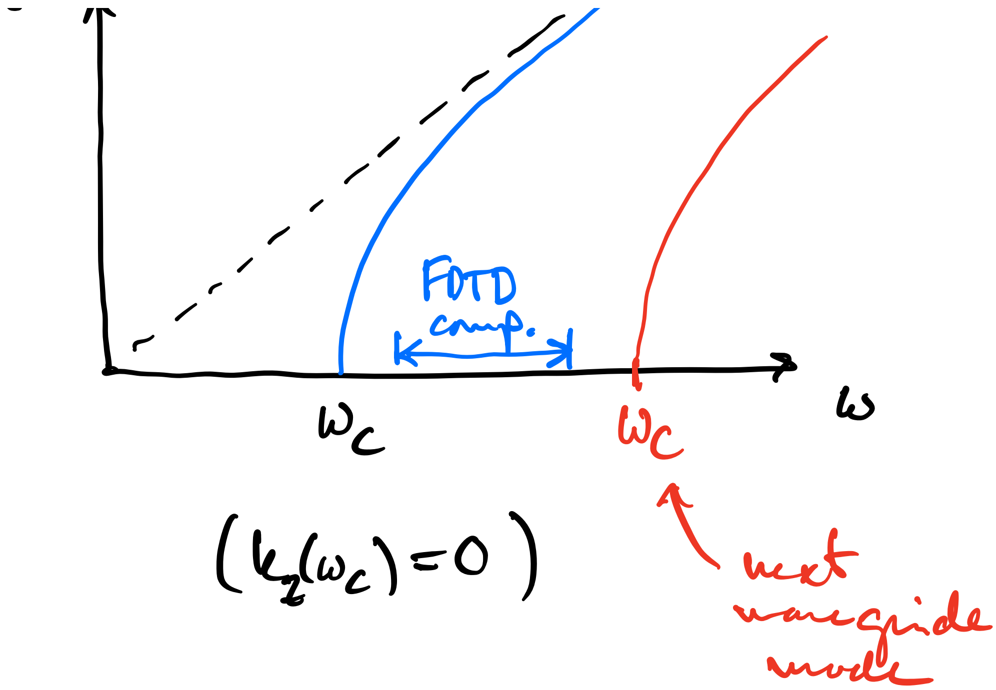
$$\begin{cases} \nabla\times\nabla\times\vec{\mathbb{E}} = \omega^2\mu_0\epsilon_0\vec{\mathbb{E}} \\ \text{with } \vec{\mathbb{E}} = \hat{y}\,2\mathbb{E}_0\cos\left(\frac{\pi x}{a}\right)e^{-jk_z z} \end{cases}$$

$$\Rightarrow \left[\left(\frac{\pi}{a}\right)^2 + k_z^2\right]\vec{\mathbb{E}} = \omega^2\mu_0\epsilon_0\vec{\mathbb{E}}$$

$$\Rightarrow \underbrace{\left[\left(\frac{\pi}{a}\right)^2 + k_z^2 - \omega^2\mu_0\epsilon_0\right]}_{=0}\vec{\mathbb{E}} = \vec{0}$$

$$\Rightarrow k_z(\omega) = \sqrt{\omega^2\mu_0\epsilon_0 - \left(\frac{\pi}{a}\right)^2}$$

k_z



$$(k_z(\omega_c) = 0)$$