Finite-differences in frequency domain

Hand-in assignment # 2 – SSY200

1 Problem description

Consider an electromagnetic plane wave that propagates towards a large flat window of glass as shown in Fig. 1. We wish to compute the reflected and transmitted wave. The glass window has the thickness $2a$.

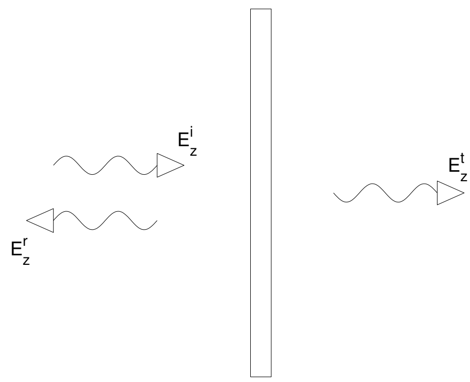

Figure 1: Glass window of thickness $2a$ with an incident field E_z^i, the reflected field E_z^r and the transmitted field E_z^t.

We have the material parameters $\epsilon(x)$, $\mu(x) = \mu_0$ and $\sigma(x)$ in the glass, where $-a \leq x \leq a$. The medium outside the window is air with $\epsilon(x) = \epsilon_0$, $\mu(x) = \mu_0$ and $\sigma = 0$ for $|x| > a$. The total electric field satisfies the differential equation:

$$-\frac{d^2 E_z(x)}{dx^2} + \mu_0 \left[j\omega\sigma(x) - \omega^2 \epsilon(x) \right] E_z(x) = 0 \tag{1}$$

2 Assignments

Here, we introduce some of the techniques used in electromagnetic scattering problems. One such technique is to formulate a boundary condition that "injects" the incoming wave by matching an expression for the incoming wave to the numerical solution in the air region. The matching is done in the air outside the scatterer, where the incoming field is known analytically. In addition, the boundary condition should be constructed such that the reflected wave (and the transmitted wave) is "received" by the same boundary condition and, thus, appear to continue to propagate in the infinite air-region outside the computational region. Again, we rely on that the reflected wave (and transmitted wave) is know analytically in the air region. Following this procedure for our 1D problem, we can do the matching at at two points $x = \pm b$ where the discretized solution in the computational domain $|x| < b$ is related to the fields outside this region by means of the boundary conditions. Here, we need to have $b > a$, so that the matching points are in the air.

2.1 Formulate boundary condition

The first task is to derive the appropriate boundary conditions at $x = -b$ and $x = b$ analytically. Here is some guidance.

Let the incident field be $E_z^i(x) = E_0 \exp(-jk_0x)$. Introduce the reflected field as $E_z^r(x) = E_r \exp(+jk_0x)$ and the transmitted field as $E_z^t(x) = E_t \exp(-jk_0x)$. You can use Eq. (1) to get $k_0 = \omega/c_0$ where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$.

What is a priori known and unknown in the expressions above? How can this information be used to formulate the appropriate boundary conditions at $x = -b$ and $x = b$, respectively? The boundary condition should involve E_z and its first derivative only. Note that b is quite arbitrary as long as it is larger than a.

2.2 Generate grid and system matrix

We use two different grids:

- `G1` The first grid is chosen such that the material interfaces $x = \pm a$ fall in-between grid points. We use the grid points $x_n = (n + \frac{1}{2})\Delta x$ with $\Delta x = 2a/N$, an even integer $N \geq 4$ and $n = -N-1, -N, \ldots, N$.

- `G2` The second grid is chosen such that the material interfaces $x = \pm a$ fall on grid points. We use the grid points $x_n = n\Delta x$ with $\Delta x = 2a/N$, an even integer $N \geq 4$ and $n = -N, -N+1, \ldots, N$.

These grids discretize a region of (roughly) length a in each air region outside the glass window, where at least two grid points are placed in the air region.

Denote the unknowns on the grid by ζ_n, i.e. $E_z(x_n) = \zeta_n$. Discretize the differential equation (1) and your boundary conditions using finite differences, both with an error that is proportional to h^2. The boundary condition involving no higher than first derivatives is best centered on the half-grid.

Using the boundary conditions and the differential equation, we have a system of linear equations $\mathbf{Az} = \mathbf{b}$ to solve, where $\mathbf{z} = [\zeta_1, \zeta_2, \ldots, \zeta_{N_{\mathrm{gp}}}]^T$ and N_{gp} is the number of grid points. Write down the matrix $\mathbf{A}$ and the right hand side $\mathbf{b}$ in the special case when $N = 4$ for the discretization `G1`. Follow the same procedure for the discretization `G2`. What are the similarities and differences when you compare the discretization `G1` and the discretization `G2`? What may the implications be and how can you handle this?

Find a way of computing the reflection and transmission coefficients from the numerical solution.

2.3 Numerical implementation

Implement your numerical algorithm in `MATLAB` for an arbitrary N and both discretizations `G1` and `G2`.

2.4 Numerical tests

Test your implementation on the case when the glass window has constant relative permittivity ϵ_r and conductivity σ. The reflection and transmission coefficient can be calculated analytically in this case and they are given by:

$$\begin{aligned} R &= \frac{(k_0^2 - k_1^2)}{\Delta} e^{j2ak_0}(e^{j4ak_1} - 1) \\ T &= \frac{k_0 k_1}{\Delta} 4 e^{j2a(k_0+k_1)} \end{aligned} \tag{2}$$

where $\Delta = (k_0 + k_1)^2 e^{j4ak_1} - (k_0 - k_1)^2$, $k_0 = \omega/c_0$ and $k_1 = \sqrt{\epsilon_r k_0^2 - j\omega\mu_0\sigma}$.

Use the thickness $a = 2$ cm in combination with the constant material parameters $\epsilon_r = 2.5$ and $\sigma = 0.02$ S/m.

For the frequency $\omega = 3 \cdot 10^9$ rad/s, compute R and T numerically on the discretization `G1` for a set of appropriately chosen values of N. Do the numerically computed values of R and T converge towards the analytical values? Which order of convergence do you find?

2.4.1 Optional problems

These problems give credit points if they are correctly solved.

5 credit points
Now repeat the above test for the discretization `G2`. Did this change the convergence properties? If so, why? By the way, how do you choose $\epsilon(\pm a)$?

5 credit points
Also, compute R and T as functions of frequency between 0.1 and 10 GHz, where you should use both the discretization `G1` and `G2` with a fixed value of N. How do the solutions compare? How does the error change with respect to the frequency? Explain your findings.

5 credit points
Compute R and T as a function of frequency between 0.1 and 10 GHz for an inhomogeneous window in the region $|x| \leq a$ with the material parameters:

$$\sigma(x) = 0.02p(x)$$
$$\epsilon(x) = \epsilon_0 \left[1 + 5p(x)\right]$$

where

$$p(x) = 1 - \left(\frac{x}{a}\right)^2$$

is a parabolic profile with $p(\pm a) = 0$.

3 Report

Compare and explain your findings in the report. It is important that you try to provide mathematical arguments to support your conclusions. Do not forget to

- describe your numerical schemes (and suitable derivations) and their implementation by `MATLAB`-program,
- results for the numerical investigations above.