Finite-difference time-domain scheme

Hand-in assignment # 2 – SSY200

1 Problem description

Waveguides and filters are important components of many complex microwave systems. Here, we consider the characteristic features for some relatively simple structures that provide a filtering functionality in waveguides. The quantities of interest is the reflection and transmission coefficient as a function of frequency. In the following, we will limit the discussion to waveguide structures with rectangular cross section and the finite-difference time-domain (FDTD) scheme [1, 2].

The FDTD model deals with the part of the waveguide that contains the filtering structure. At each of the two ends of the filter, a shorter section of a rectangular waveguide is attached and truncated by a port for computational modeling purposes. (The physical waveguide would normally extend beyond the ports but that part is not included in the computational model considered here.)

1.1 Modal representation for a rectangular waveguide

In an air-filled rectangular waveguide with the transverse dimensions L_x and L_y, we can decompose the electric and magnetic fields into *transverse electric* (TE_{mn}) and *transverse magnetic* (TM_{mn}) modes, see Ref. [3] for a detailed discussion. Each mode has its own propagation constant

$$\gamma_{mn} = \sqrt{h_{mn}^2 - \left(\frac{\omega}{c}\right)^2} \tag{1}$$

where h_{mn}^2 are the eigenvalues of the transverse problem for H_z (TE case) or E_z (TM case), i.e.

$$h_{mn}^2 = \left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2. \tag{2}$$

For TE modes, m and n are non-negative integers and also $m + n > 0$. For TM modes, both m and n are positive integers.

Numbering the modes from 1 to ∞, we can express the electric field in the waveguide as a superposition of both TE and TM modes as

$$\vec{E}(x, y, z, t) = \sum_{n=1}^{\infty} V_m(z, t)\ \vec{e}_m(x, y) \tag{3}$$

where $V_m(z, t)$ is the modal amplitude, or *voltage*, of mode m (which can be either a TE or a TM mode) and $\vec{e}_m(x, y)$ is its modal field. The modal field, $\vec{e}_m(x, y)$, for the TE_{10} mode is shown in Fig. 1.

In the following, we will consider situations where the frequency range of interest and the dimensions (L_x, L_y) of the waveguide are chosen so that γ_{mn} is real for all

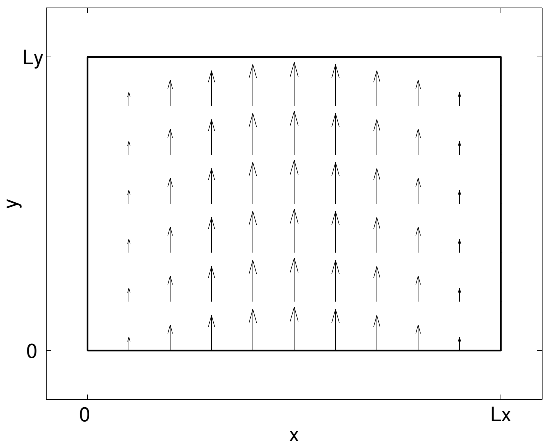

Figure 1: Modal field of the TE_{10} mode.

modes except one, namely the TE_{10} mode. Consequently, only the TE_{10} mode will propagate and exist far away from any source or irregularity in the waveguide, since the other modes are evanescent and decay exponentially according to $V_m \propto \exp(-\gamma z)$.

1.2 Computation of the S-parameters

A filter can be characterized in terms of its reflection and transmission coefficient and, in a more general setting, these are often referred to as the scattering parameters or simply the S-parameters. Figure 2 shows a rectangular waveguide (without the filtering structure) that is truncated at two ports for computational modeling purposes. The S-parameters can be computed given the relation between the amplitudes of the TE_{10} mode at the ports: (i) an incident wave is launched at one port, (ii) the reflected wave is recorded at the same port, and (iii) the transmitted wave is recorded at the other port.

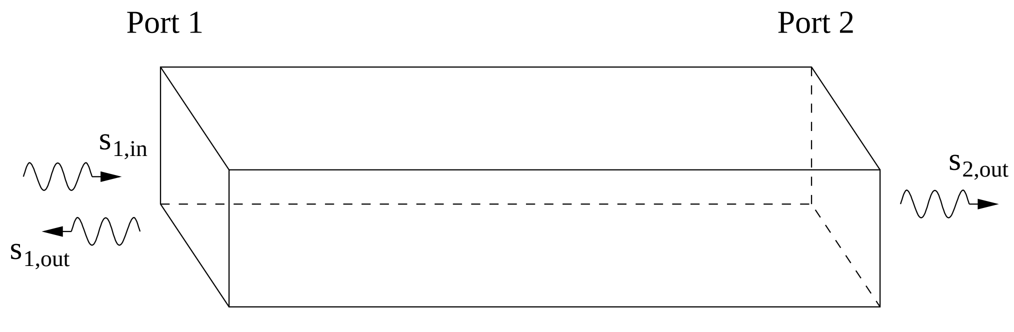

Figure 2: Illustration of incoming and outgoing waves.

Let $s_{1,\text{in}}(t)$ be the amplitude of the incoming TE_{10} wave at port 1 and and let $s_{1,\text{out}}(t)$ and $s_{2,\text{out}}(t)$ be the amplitudes of the outgoing TE_{10} waves at port 1 and 2, respectively. The Fourier transform of these signals gives $S_{1,\text{in}}(\omega)$, $S_{1,\text{out}}(\omega)$ and $S_{2,\text{out}}(\omega)$. The relation between the amplitudes at the two ports are usually described by the so-called S-parameters:

$$S_{11}(\omega) = \frac{S_{1,\mathrm{out}}(\omega)}{S_{1,\mathrm{in}}(\omega)} \tag{4}$$

$$S_{21}(\omega) = \frac{S_{2,\mathrm{out}}(\omega)}{S_{1,\mathrm{in}}(\omega)} \tag{5}$$

The scattering parameter S_{11} is recognized as the reflection coefficient and S_{21} as the transmission coefficient. A further extension to an n-port network is rather straightforward and for such cases it is convenient to represent the S-parameters in matrix form, an $n \times n$ scattering matrix $\mathbf{S}$ with the elements S_{ij}.

1.3 Numerical modeling

The interior of the waveguide is discretized by an FDTD grid [1, 2]. A wave can be launched at one of the ports and then propagated through the waveguide by means of Maxwell's equations represented by the FDTD scheme applied to the grid in-between the ports. Consequently, a filtering structure can be modeled in detail by the FDTD scheme and its reflection and transmission coefficient be computed from the fields at the ports.

The special type of boundary condition that is required at the ports is already implemented in the `MATLAB` program provided as a starting point for the tasks below. The algorithm is briefly summarized in Appendix A.

2 Assignments

You are given a `MATLAB` program that should be used as a starting point when you work with the tasks listed below. The implementation provided allocates the required memory areas for the electric and magnetic fields for an empty waveguide with the dimensions:

$$L_x = 40.0 \text{ mm},\ L_y = 22.5 \text{ mm and } L_z = 160.0 \text{ mm}. \tag{6}$$

It also contains parts that deal with an incoming signal, a Gaussian-modulated sinusoidal pulse, that contains energy in the frequency interval from 3.9 to 6.5 GHz. The waveguide ports (that impose the incident wave and absorbs the reflected and transmitted wave) are located at $z = 0$ mm and $z = 160.0$ mm. Given this program, solve the problems that follow below, where some are optional as indicated in the text.

2.1 Numerical implementation

Implement the update loops for Faraday's and Ampère's law according to the FDTD scheme in 3D for an empty rectangular waveguide.

2.2 Numerical tests

What is the expected reflected $s_{1,\text{out}}(t)$ and transmitted $s_{2,\text{out}}(t)$ solution for an empty waveguide given the Gaussian excitation pulse? Test your code and see if the result is what you expected. What is the cut-off frequency of the TE_{10} mode? Which mode has the second lowest cut-off frequency and what frequency is that?

Implement a post-processing step that transforms the time-domain scattering amplitudes $s_{1,\text{out}}(t)$ and $s_{2,\text{out}}(t)$ to their corresponding frequency-domain quantities and provide code that evaluates the scattering parameters (4) and (5). Verify that your implementation works as expected for the empty waveguide. You can calculate the analytical scattering parameters (4) and (5), which makes a careful comparison feasible. Comment on your findings by an interpretation of the numerical errors.

2.2.1 Optional problems

These problems give credit points if they are correctly solved.

5 credit points
For the first optional problem, change the `MATLAB` program so that you can analyze a waveguide that has a somewhat more narrow mid-section as shown in Fig. 3. The dimensions are $a = w = 4$ cm, $b = 6$ cm and $d = 3$ cm. The walls are perfectly conducting and the geometry is independent on the y-coordinate. Compute $|S_{11}(\omega)|$ and $|S_{21}(\omega)|$ for this modified geometry. Comment on your findings and provide an explanation to the transfer function that you find.

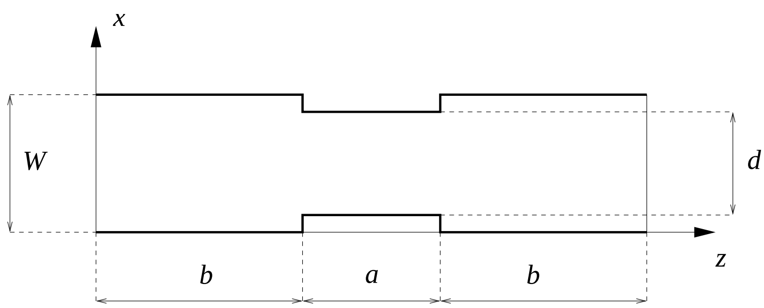

Figure 3: Waveguide with a narrow mid-section.

10 credit points
The more challenging optional problem is to model a lossy dielectric block placed "on the floor" of the waveguide as shown in Fig. 4, which contains a top and side view of the dielectric block indicated by the shaded region. The dimensions in Fig. 4 are $a = 2$ cm, $b = 7$ cm, $d = 2$ cm and $h = 1$ cm. (Note that the dielectric block in the figure is not to scale.) The relative permittivity of the dielectric block is 5 and its conductivity is 0.07 S/m. Here, it is necessary to modify Ampère's law in the implementation of the FDTD update scheme to incorporate an inhomogeneous dielectric together with the eddy currents that result from a non-zero conductivity. Note that Ampère's law is evaluated at half-integer time-steps (i.e. $n+1/2$) and, thus,

the eddy currents are best evaluated according to the (semi-implicit) approximation

$$\vec{J}(\vec{r}, t = (n + 1/2)\Delta t) = \sigma(\vec{r}) \frac{\vec{E}(\vec{r}, t = (n + 1)\Delta t) + \vec{E}(\vec{r}, t = n\Delta t)}{2}$$

which produces a numerically stable and accurate time-stepping scheme for all values of σ from zero to infinity. How do you deal with the update of the electric field for this problem? Report on your new (explicit) update formulas that give the electric field according to Ampère's law. In particular, comment on how the electric field tangential to the surface between the dielectric and the air is updated. Compute $|S_{11}(\omega)|$ and $|S_{21}(\omega)|$ for this modified problem. How much power is reflected and transmitted as a function of frequency? How much power is absorbed in the dielectric medium as a function of frequency? Can you use the visualized fields to interpret your findings?

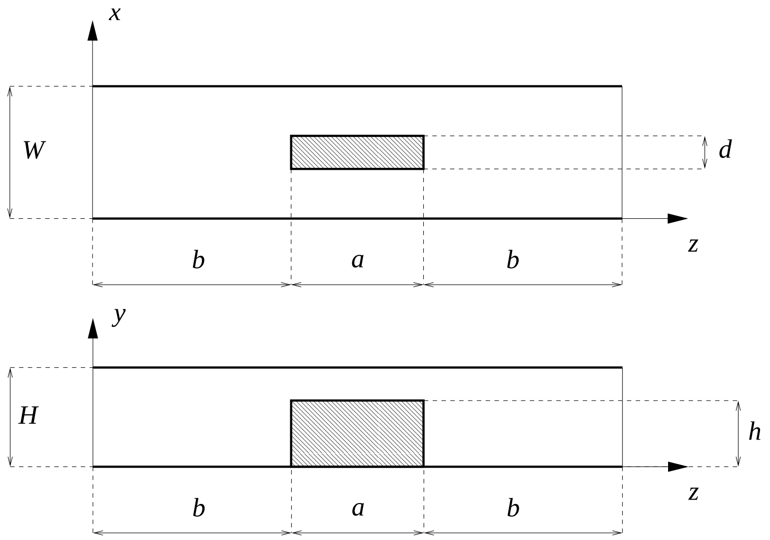

Figure 4: Waveguide with a lossy dielectric block placed "on the floor".

A Waveguide ports

The algorithm is briefly summarized as follows (see Ref. [4] for further details):

- At each time step, n, we extract the transverse electric field one cell away from the port boundary. Let's denote this with $\vec{E}_t|^n_{p,q,N_z-1}$. Clearly, this field can be represented as a superposition of waveguide modes that propagate along both directions of the waveguide. Below, we consider for simplicity a port that does not have an incident wave.

- With this result, we can compute the voltages $V_m|^n_{N_z-1}$ of the different modes[1] m, *one cell away from* the boundary.

$$V_m|^n_{N_z-1} = \sum_{p,q} \Delta x \, \Delta y \; \vec{E}_t|^n_{p,q,N_z-1} \cdot \vec{e}_m|_{p,q} \tag{7}$$

[1]The number of required modes depends on how close to any discontinuities the ports are placed; the further away from any discontinuities, the more decayed are the higher order modes.

Since it is known that there is no incident mode at this port, the decomposed field only consists of waveguide modes that are propagating out from the computational domain. (For a port with an incident field, this could easily be compensated for since the user knows what incident field is used and, thus, the incident field can be subtracted from the total field to yield the reflected field.)

- Each mode can be modeled by a one-dimensional wave equation

$$\frac{\partial^2 V_m}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2 V_m}{\partial t^2} - h_m^2 V_m = 0, \tag{8}$$

which can be discretized as

$$V_m|_r^{n+1} = AV_m|_r^n + B(V_m|_{r+1}^n + V_m|_{r-1}^n) - V_m|_r^{n-1} \tag{9}$$

where $V_m|_r^n \equiv V_m(r\Delta z, n\Delta t)$ and

$$\begin{aligned} B &= \left(\frac{c_0\Delta t}{\Delta z}\right)^2 \\ A &= 2 - 2B - (c_0\Delta t\, h_m)^2 . \end{aligned}$$

The impulse response $I_m|^n = V_m|_1^n$ for this one-dimensional wave equation can be computed from Eq. (9). Provided this impulse response, we can use a one dimensional convolution to compute the voltages *on* the boundary that coincides with the port:

$$V_m|_{N_z}^n = V_m|_{N_z-1}^n * I_m|^n = \sum_{j=1}^{n} V_m|_{N_z-1}^{n-j} \cdot I_m|^j \tag{10}$$

- Now, we know the modal voltages on the port boundary. The total electric field on the boundary is a linear combination of the modal fields

$$\vec{E}|_{p,q,N_z}^n = \sum_{m=1}^{M} V_m|_{N_z}^n \; \vec{e}_m|_{p,q}, \tag{11}$$

and this solution is explicitly written into the FDTD grid before the next update of the interior grid points that are located inside the computational domain.

References

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, May 1966.

[2] A. Taflove and S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method (2nd Edition)*. Norwood, MA: Artech House, 2000.

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[4] F. Alimenti, P. Mezzanotte, L. Roselli, and R. Sorrentino, "A revised formulation of model absorbing and matched modal source boundary conditions for the efficient FDTD analysis of waveguide structures," *IEEE Trans. Microwave Theory Tech.*, vol. 48, pp. 50–59, January 2000.