Formelsamling

Linjära System
och
Transformer

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Institutionen för Signaler och System

Formulas for TMA982 Linjära System och Transformer

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Contents

Preface

The definitions and conventions used in this document comply with those used in Oppenheim and Willsky [1]. Most of the material originates from [1] and [2].

1 Terminology

LTI Linear Time-Invariant

ROC Region Of Convergence

BIBO Bounded Input Bounded Output

2 Definitions

2.1 Linearity

A continuous-time system is linear if:

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

A discrete-time system is linear if:

$$ax_1[n] + bx_2[n] \to ay_1[n] + by_2[n]$$

2.2 Causality

A system is said to be causal if the output of the system at any time depends only on present and past inputs.
A continuous-time LTI system is causal if its impulse response satisfies

$$h(t) = 0, \text{ for } t < 0$$

A discrete-time LTI system is causal if its impulse response satisfies

$$h[n] = 0, \text{ for } n < 0$$

2.3 Time-invariance

A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal:

$$x(t) \to y(t) \Rightarrow x(t - t_0) \to y(t - t_0)$$

$$x[n] \to y[n] \Rightarrow x[n - n_0] \to y[n - n_0]$$

2.4 Periodicity

A continuous-time signal is periodic with period $T_p > 0$ if and only if

$$x(t + T_p) = x(t), \text{ for all } t$$

A discrete-time signal is periodic with period $N > 0$ if and only if

$$x[n + N] = x[n], \text{ for all } n$$

2.5 Evenness and oddness

A continuous-time signal is said to be even if

$$x(-t) = x(t)$$

A discrete-time signal is said to be even if

$$x[-n] = x[n]$$

A continuous-time signal is said to be odd if

$$x(-t) = -x(t)$$

A discrete-time signal is said to be odd if

$$x[-n] = -x[n]$$

Any signal can be broken into a sum of one even and one odd signal ($Ev\{\cdot\}$ even part and $Od\{\cdot\}$ odd part):

$$Ev\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$$

$$Ev\{x[n]\} = \frac{1}{2}(x[n] + x[-n])$$

$$Od\{x(t)\} = \frac{1}{2}(x(t) - x(-t))$$

$$Od\{x[n]\} = \frac{1}{2}(x[n] - x[-n])$$

2.6 Stability

An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces a bounded output.
A continuous-time LTI system is stable if and only if its impulse response is *absolutely integrable*, i.e.

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

A discrete-time LTI system is stable if and only if its impulse response is *absolutely summable*, i.e.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

3 Functions

3.1 The continuous-time unit impulse function

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

3.2 The continuous-time unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

3.3 The discrete-time unit impulse sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

3.4 The discrete-time unit step sequence

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

4 Convolution

4.1 Continuous-time convolution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

4.2 Discrete-time convolution

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

4.3 Circular convolution

$$x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2[((n-k))_N], \qquad n = 0, 1, \ldots, N-1$$

$$x[((n-k))_N] = x[(n-k) \bmod N]$$

4.4 Properties of convolution

Property	Relation
Linearity	$(ax_1(t) + bx_2(t)) * h(t) = a(x_1(t) * h(t)) + b(x_2(t) * h(t))$
	$(ax_1[n] + bx_2[n]) * h[n] = a(x_1[n] * h[n]) + b(x_2[n] * h[n])$
Commutativity	$x(t) * h(t) = h(t) * x(t)$
	$x[n] * h[n] = h[n] * x[n]$
Distributivity	$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$
	$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
Associativity	$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
	$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

5 Correlation

5.1 Continuous-time correlation

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau, \text{ cross-correlation}$$

$$\phi_{xx}(t) = \int_{-\infty}^{\infty} x(t+\tau)x(\tau)d\tau, \text{ autocorrelation}$$

5.2 Discrete-time correlation

$$\phi_{xy}[n] = \sum_{m=-\infty}^{\infty} x[m+n]y[m], \text{ cross-correlation}$$

$$\phi_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x[m], \text{ autocorrelation}$$

6 Energy and power

6.1 Continuous-time signal energy

$$E_{\infty} = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

6.2 Discrete-time signal energy

$$E_{\infty} = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

6.3 Continuous-time signal power

$$P_{\infty} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

periodic signal with period T:

$$P_{\infty} = \frac{1}{T} \int_{T} |x(t)|^2 dt$$

6.4 Discrete-time signal power

$$P_{\infty} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

periodic signal with period N:

$$P_{\infty} = \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$$

7 Sampling

7.1 Spectrum of sampled signals

Let $x_d[n] = x_c(nT)$, then the Fourier transform of the discrete-time signal, $X_d(e^{j\Omega})$, is related to the Fourier transform of the analog signal, $X_c(j\omega)$, as

$$X_d(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)$$

7.2 Sampling theorem

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M$$

where

$$\omega_s = \frac{2\pi}{T}$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

7.3 Reconstruction

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c(t - nT))}{\omega_c(t - nT)}$$

with

$$\omega_c = \omega_s/2$$

8 General relations

8.1 Geometric series

finite:

$$\sum_{k=0}^{n-1} x^k = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n, & x = 1 \end{cases}$$

infinite:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, |x| < 1$$

8.2 Euler's formulas

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

9 The Laplace transform

9.1 Definition

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

9.2 Inverse equation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

9.3 Properties of the Laplace transform

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \bigcap R_2$
Time Shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e. s is in the ROC if $s - s_0$ is in R)
Time Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e. s in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \bigcap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R\bigcap\{\mathrm{Re}\{s\} > 0\}$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at t=0, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

9.4 Laplace transforms of elementary functions

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathrm{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathrm{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathrm{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
12	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
13	$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -\alpha$
14	$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	All s
16	$u_{-n}(t) = u(t) * \cdots * u(t)$, n times	$\frac{1}{s^n}$	$\mathrm{Re}\{s\} > 0$

10 The Z-transform

10.1 Definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

10.2 Inverse equation

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

10.3 Properties of the Z-transform

Property	Signal	Z-Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \bigcap R_2$
Time Shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of $z = 0$
Scaling in the z-Domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
	$a^n x[n]$	$X\left(a^{-1}z\right)$	Scaled version of R (i.e. $\|a\|R =$ the set of points $\{\|a\|z\}$ for z in R)
Time Reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e. $R^{-1} =$ the set of points z^{-1}, where z is in R)
Time Expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$, r integer	$X(z^k)$	$R^{1/k}$ (i.e. the set of points $z^{1/k}$, where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \bigcap R_2$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least $R \bigcap \{\|z\| > 0\}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least $R \bigcap \{\|z\| > 1\}$
Differentiation in the z-Domain	$nx[n]$	$-z\frac{d}{dz}X(z)$	R

Initial Value Theorem

If $x[n] = 0$ for $n < 0$ then

$$x[0] = \lim_{z \to \infty} X(z)$$

10.4 Some common Z-transform pairs

Transform pair	Signal	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$\delta[n-m]$	z^{-m}	All z, except for $\begin{cases} z=0, m>0 \\ z=\infty, m<0 \end{cases}$
5	$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6	$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7	$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8	$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11	$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12	$r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

11 The Fourier series and transforms

11.1 The continuous-time Fourier series, CTFS

11.1.1 Definition

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

11.1.2 Power density spectrum

$$|a_k|^2$$

11.1.3 Convergence criterion

The Dirichlet conditions are:
Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_T |x(t)|dt < \infty.$$

Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.
Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

11.1.4 Properties of continuous-time Fourier series

Property	Periodic Signal	Fourier Series Coefficients
	$x(t)$ periodic T, fundamental frequency $\omega_0 = \frac{2\pi}{T}$	a_k
	$y(t)$ periodic T, fundamental frequency $\omega_0 = \frac{2\pi}{T}$	b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Frequency Shifting	$e^{jM\omega_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t)$, $\alpha > 0$	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$ (finite and periodic only if $a_0 = 0$)	$\frac{1}{jk\omega_0} a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathrm{Ev}\{x(t)\} \\ x_o(t) = \mathrm{Od}\{x(t)\} \end{cases}$, $x(t)$ real	$\begin{cases} \mathrm{Re}\{a_k\} \\ j\,\mathrm{Im}\{a_k\} \end{cases}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

11.2 The continuous-time Fourier transform, CTFT

11.2.1 Definition

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

11.2.2 Energy density spectrum

$$|X(j\omega)|^2$$

11.2.3 Convergence criterion

The Dirichlet conditions are:
Condition 1. $x(t)$ must be absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty.$$

Condition 2. $x(t)$ have a finite number of maxima and minima within any finite interval.
Condition 3. $x(t)$ have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

11.2.4 Properties of the continuous-time Fourier transform

Property	Aperiodic Signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathrm{Re}\{X(j\omega)\}=\mathrm{Re}\{X(-j\omega)\} \\ \mathrm{Im}\{X(j\omega)\}=-\mathrm{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \angle X(j\omega)=-\angle X(-j\omega) \end{cases}$
Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t)=\mathrm{Ev}\{x(t)\} \\ x_o(t)=\mathrm{Od}\{x(t)\} \end{cases}$, $x(t)$ real	$\begin{cases} \mathrm{Re}\{X(j\omega)\} \\ j\,\mathrm{Im}\{X(j\omega)\} \end{cases}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

11.3 Basic continuous-time Fourier transform pairs

Transform pair	Signal	Fourier transform	Fourier series coefficients (if periodic)
1	$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
2	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$\begin{cases} a_1 = 1 \\ a_k = 0, \text{ otherwise} \end{cases}$
3	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$\begin{cases} a_1 = a_{-1} = \frac{1}{2} \\ a_k = 0, \text{ otherwise} \end{cases}$
4	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$\begin{cases} a_1 = -a_{-1} = \frac{1}{2j} \\ a_k = 0, \text{ otherwise} \end{cases}$
5	1	$2\pi\delta(\omega)$	$\begin{cases} a_0 = 1 \\ a_k = 0, \text{ otherwise} \end{cases}$
6	$x(t) = \begin{cases} 1, \lvert t\rvert < T_1 \\ 0, T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ periodic T	$\sum_{k=-\infty}^{\infty} 2\frac{\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$	$\frac{\sin(k\omega_0 T_1)}{k\pi}$
7	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}\forall k$
8	$x(t) = \begin{cases} 1, \lvert t\rvert < T_1 \\ 0, \lvert t\rvert > t_1 \end{cases}$	$\frac{2\sin(\omega T_1)}{\omega}$	–
9	$\frac{\sin(Wt)}{\pi t}$	$X(j\omega) = \begin{cases} 1, \lvert\omega\rvert < W \\ 0, \lvert\omega\rvert > W \end{cases}$	–
10	$\delta(t)$	1	–
11	$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	–
12	$\delta(t-t_0)$	$e^{-j\omega t_0}$	–
13	$e^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{1}{a+j\omega}$	–
14	$te^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$	–
15	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\mathrm{Re}\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$	–

11.4 The discrete-time Fourier series, DTFS

11.4.1 Definition

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

11.4.2 Power density spectrum

$$|a_k|^2$$

11.4.3 Properties of the discrete-time Fourier series

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ periodic N, fundamental frequency $\omega_0 = \frac{2\pi}{N}$	a_k periodic N
	$y[n]$ periodic N, fundamental frequency $\omega_0 = \frac{2\pi}{N}$	b_k periodic N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk\omega_0 n_0}$
Frequency Shifting	$e^{jM\omega_0 n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], n \text{ multiple of } m \\ 0, \text{otherwise} \end{cases}$, periodic mN	$\frac{1}{m} a_k$ periodic mN
Periodic Convolution	$\sum_{r=<N>} x[r]y[n-r]$	$N a_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk\omega_0}) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite and periodic only if $a_0 = 0$)	$\frac{1}{1-e^{-jk\omega_0}} a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathrm{Ev}\{x[n]\} \\ x_o[n] = \mathrm{Od}\{x[n]\} \end{cases}$, $x[n]$ real	$\begin{cases} \mathrm{Re}\{a_k\} \\ j\,\mathrm{Im}\{a_k\} \end{cases}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

11.5 The discrete-time Fourier transform, DTFT

11.5.1 Definition

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

11.5.2 Energy density spectrum

$$|X(e^{j\omega})|^2$$

11.5.3 Properties of the discrete-time Fourier transform

Property	Aperiodic Signal	Fourier transform
	$x[n]$	$X(e^{j\omega})$ periodic 2π
	$y[n]$	$Y(e^{j\omega})$ periodic 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], n \text{ multiple of } m \\ 0, \text{ otherwise} \end{cases}$	$X(e^{jm\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differentiation in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{d}{d\omega}X(e^{j\omega})$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\} \\ \mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}) \end{cases}$
Real and Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
Real and Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathrm{Ev}\{x[n]\} \\ x_o[n] = \mathrm{Od}\{x[n]\} \end{cases}$, $x[n]$ real	$\begin{cases} \mathrm{Re}\{X(e^{j\omega})\} \\ j\,\mathrm{Im}\{X(e^{j\omega})\} \end{cases}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

11.6 Basic discrete-time Fourier transform pairs

Transform pair	Signal	Fourier transform	Fourier series coefficients (if periodic)
1	$\sum_{k=<N>} a_k e^{jk\omega_0 n}$, $\omega_0 = \frac{2\pi}{N}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
2	$e^{j\omega_0 n}$, $\frac{\omega_0}{2\pi} = \frac{m}{N}$ rational	$2\pi\delta(\omega-\omega_0)$ periodic 2π	$a_k = \begin{cases} 1, k=m \\ 0, \text{otherwise} \end{cases}$ periodic N
3	$e^{j\omega_0 n}$, $\frac{\omega_0}{2\pi}$ irrational (aperiodic)	$2\pi\delta(\omega-\omega_0)$ periodic 2π	–
4	$\cos(\omega_0 n)$, $\frac{\omega_0}{2\pi} = \frac{m}{N}$ rational	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ periodic 2π	$a_k = \begin{cases} \frac{1}{2}, k=\pm m \\ 0, \text{otherwise} \end{cases}$ periodic N
5	$\cos(\omega_0 n)$, $\frac{\omega_0}{2\pi}$ irrational (aperiodic)	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ periodic 2π	–
6	$\sin(\omega_0 n)$, $\frac{\omega_0}{2\pi} = \frac{m}{N}$ rational	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$ periodic 2π	$a_k = \begin{cases} \frac{1}{2j}, k=m \\ -\frac{1}{2j}, k=-m \\ 0, \text{otherwise} \end{cases}$ periodic N
7	$\sin(\omega_0 n)$, $\frac{\omega_0}{2\pi}$ irrational (aperiodic)	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$ periodic 2π	–
8	1	$2\pi\delta(\omega)$ periodic 2π	$a_k = \begin{cases} 1, k=0 \\ 0, \text{otherwise} \end{cases}$ periodic N
9	$x[n] = \begin{cases} 1, \lvert n\rvert \le N_1 \\ 0, N_1 < \lvert n\rvert \le \frac{N}{2} \end{cases}$ periodic N	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \begin{cases} \frac{\sin[(2\pi k/N)(N_1+1/2)]}{N\sin(\pi k/N)}, k \neq 0 \\ \frac{2N_1+1}{N}, k=0 \end{cases}$ periodic N
10	$\sum_{k=-\infty}^{\infty} \delta[n-kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N} \forall k$
11	$x[n] = \begin{cases} 1, \lvert n\rvert \le N_1 \\ 0, \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1+1/2)]}{\sin(\omega/2)}$	–
12	$\frac{\sin(Wn)}{\pi n}$, $0 < W < \pi$	$X(e^{j\omega}) = \begin{cases} 1, \lvert\omega\rvert \le W \\ 0, W < \lvert\omega\rvert \le \pi \end{cases}$, periodic 2π	–
13	$\delta[n]$	1	–
14	$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$, periodic 2π	–
15	$\delta[n-n_0]$	$e^{-j\omega n_0}$	–
16	$a^n u[n], \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$	–
17	$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$	–
18	$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$	–

11.7 The discrete Fourier transform, DFT

11.7.1 Definition

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, k = 0, 1, \ldots, N-1$$

11.7.2 Inverse equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}, n = 0, 1, \ldots, N-1$$

11.7.3 Properties of the discrete Fourier transform

All DFTs in the table below are N-point DFTs. Thus, all signals in the left-hand side column are zero for $n < 0$ and $n > N-1$. Likewise, the transforms in the right-hand side column are zero for $k < 0$ and $k > N-1$

Property	N-Point Signal	N-Point DFT
	$x[n]$, $x_1[n]$, $x_2[n]$	$X[k]$, $X_1[k]$, $X_2[k]$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1[k] + a_2 X_2[k]$
Duality	$X[n]$	$Nx[((-k))_N]$
Circular Time-Shift	$x[((n-m))_N]$	$X[k]e^{-j2\pi\frac{m}{N}k}$
Modulation	$x[n]e^{j2\pi\frac{m}{N}k}$	$X[((k-m))_N]$
Circular Folding	$x[((-n))_N]$	$X[((-k))_N]$
Circular Convolution	$\sum_{k=0}^{N-1} x_1[k] x_2[((n-k))_N]$	$X_1[k]X_2[k]$
Conjugation	$x^*[n]$	$X^*[((-k))_N]$
Conjugate Symmetry	$x_e[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\text{Re}\{X[k]\}$
Conjugate Antisymmetry	$x_o[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\,\text{Im}\{X[k]\}$
The following properties apply when $x[n]$ is real		
Symmetries	any real signal $x[n]$	$X[k] = X^*[((-k))_N]$ $\text{Re}\{X[k]\} = \text{Re}\{X[((-k))_N]\}$ $\text{Im}\{X[k]\} = -\text{Im}\{X[((-k))_N]\}$ $\lvert X[k]\rvert = \lvert X[((-k))_N]\rvert$ $\arg\{X[k]\} = -\arg\{X[((-k))_N]\}$
Even Symmetry	$x_e[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\text{Re}\{X[k]\}$
Odd Symmetry	$x_o[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\,\text{Im}\{X[k]\}$

11.8 Summary of Fourier series and transform expressions

	Fourier series		Fourier transform	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Continuous time	CTFS		CTFT	
	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $\omega_0 = \frac{2\pi}{T}$	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
	continuous, periodic	discrete, aperiodic	continuous, aperiodic	continuous, aperiodic
Discrete time	DTFS		DTFT	
	$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$, $\omega_0 = \frac{2\pi}{N}$	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
	discrete, periodic	discrete, periodic	discrete, aperiodic	continuous, periodic

12 Continuous-time LTI filter

12.1 Frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = |H(j\omega)|e^{j\angle X(j\omega)}$$

12.2 Magnitude response

$$|H(j\omega)|$$

12.3 Phase response

$$\angle H(j\omega)$$

12.4 Phase delay

$$\tau_p(\omega) = -\frac{\angle H(j\omega)}{\omega}$$

12.5 Group delay

$$\tau_g(\omega) = -\frac{d}{d\omega}\angle H(j\omega)$$

13 Discrete-time LTI filter

13.1 Frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

13.2 Magnitude response

$$|H(e^{j\omega})|$$

13.3 Phase response

$$\angle H(e^{j\omega})$$

13.4 Phase delay

$$\tau_p(\omega) = -\frac{\angle H(e^{j\omega})}{\omega}$$

13.5 Group delay

$$\tau_g(\omega) = -\frac{d}{d\omega}\angle H(e^{j\omega})$$

14 Window functions

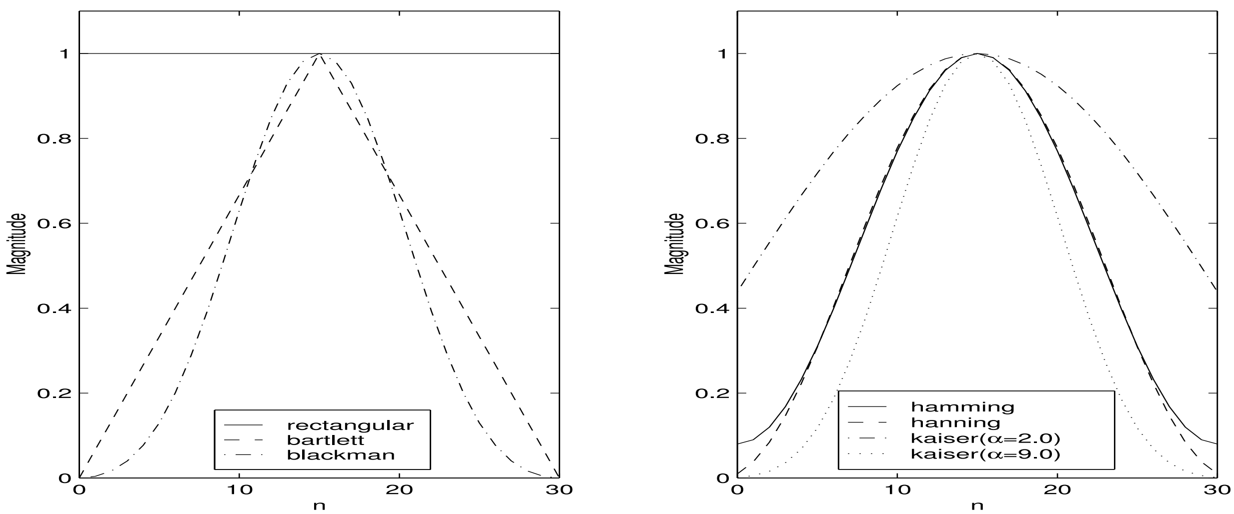
14.1 Definitions

Name of window	Time-domain sequence, $h[n]$, $0 \le n \le M-1$
Rectangular	1
Bartlett (triangular)	$1 - 2\frac{\left\lvert n - \frac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right)$
Hamming	$0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$
Hanning	$\frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M-1}\right)\right]$
Kaiser(α)	$\frac{I_0\left(\alpha\left[\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2\right]^{\frac{1}{2}}\right)}{I_0\left(\alpha\frac{M-1}{2}\right)}$

$I_0(\cdot)$ is the zero:th order Bessel function.

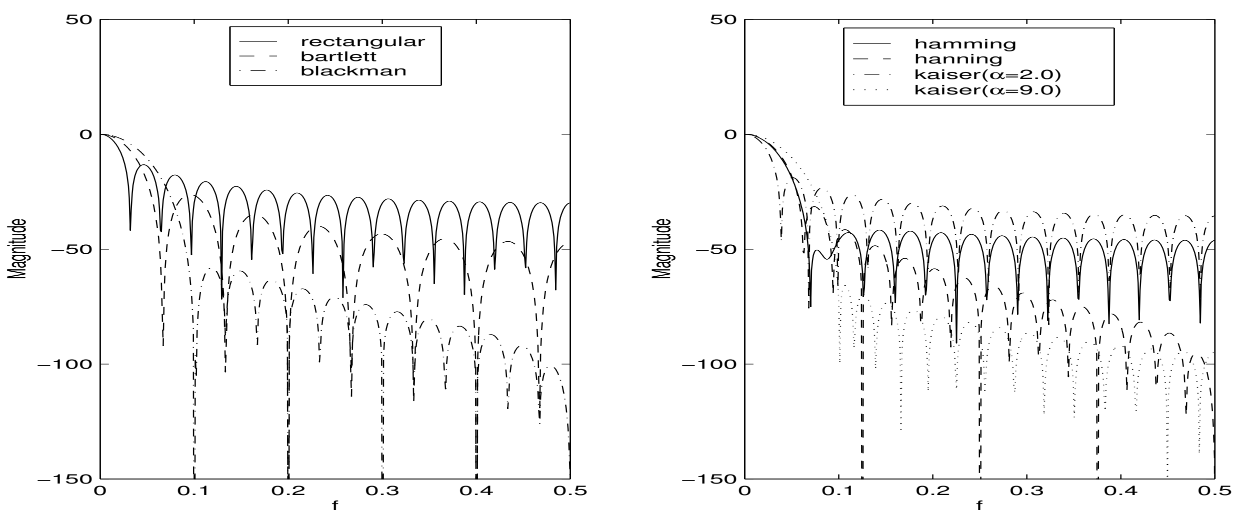
14.2 Time plots of some window functions

In the plots, $M = 31$ is assumed.



14.3 Frequency response plots of some window functions

In the plots, $M = 31$ is assumed.



14.4 Frequency-domain characteristics of some window functions

Name of window	Approximate transition width of main lobe (M window length)	Peak sidelobe (dB)
Rectangular	$\frac{4\pi}{M}$	-13
Bartlett	$\frac{8\pi}{M}$	-27
Hanning	$\frac{8\pi}{M}$	-32
Hamming	$\frac{8\pi}{M}$	-43
Blackman	$\frac{12\pi}{M}$	-58

15 Own notes

On this and the next page you have extra space for your own notes.

Own notes cont.

16 References

[1] Alan V. Oppenheim and Alan S. Willsky. *Signals and Systems.* Prentice-Hall, Upper Saddle River, New Jersey 07458, USA, second edition, 1997.

[2] John G. Proakis and Dimitris G. Manolakis. *Digital Signal Processing.* Prentice-Hall, Upper Saddle River, New Jersey 07458, USA, third edition, 1996.