CHAPTER 9

Discrete-Time Signals

A basic understanding of discrete-time signal processing is essential in the design of most modern analog systems. For example, discrete-time signal processing is heavily used in the design and analysis of oversampling analog-to-digital (A/D) and digital-to-analog (D/A) converters used in digital audio and instrumentation applications. Also, a discrete-time filtering technique known as switched-capacitor filtering is probably the most popular approach for realizing fully integrated analog filters. Switched-capacitor filters are in the class of analog filters since voltage levels in these filters remain continuous. In other words, switched-capacitor filters operate and are analyzed using discrete-time steps but involve no A/D or D/A converters. This chapter presents some basic concepts of discrete-time signals and filters.

9.1 OVERVIEW OF SOME SIGNAL SPECTRA

Consider the spectra of sampled and continuous-time signals in the block diagram systems shown in Fig. 9.1, where it is assumed that the continuous-time signal, $x_c(t)$, is band limited through the use of an anti-aliasing filter (not shown). *DSP* refers to *Discrete-time signal processing,* which may be accomplished using fully digital processing or discrete-time analog circuits such as switched-capacitor filters. Some

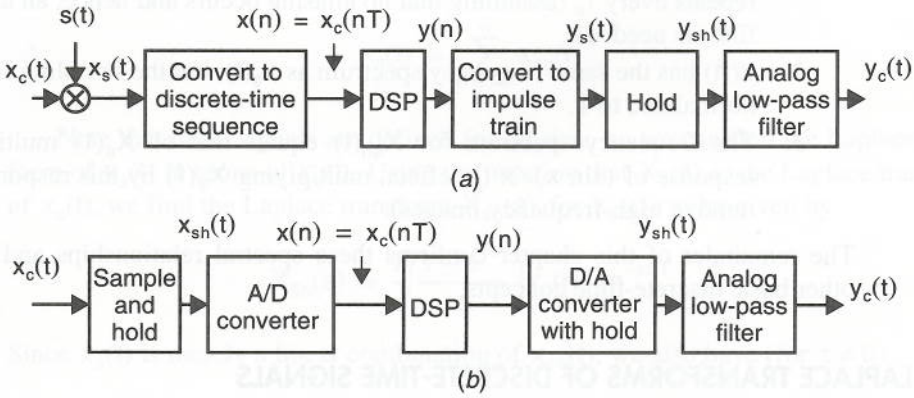

Fig. 9.1 Performing DSP on analog signals. (*a*) Conceptual realization, and (*b*) typical physical realization.

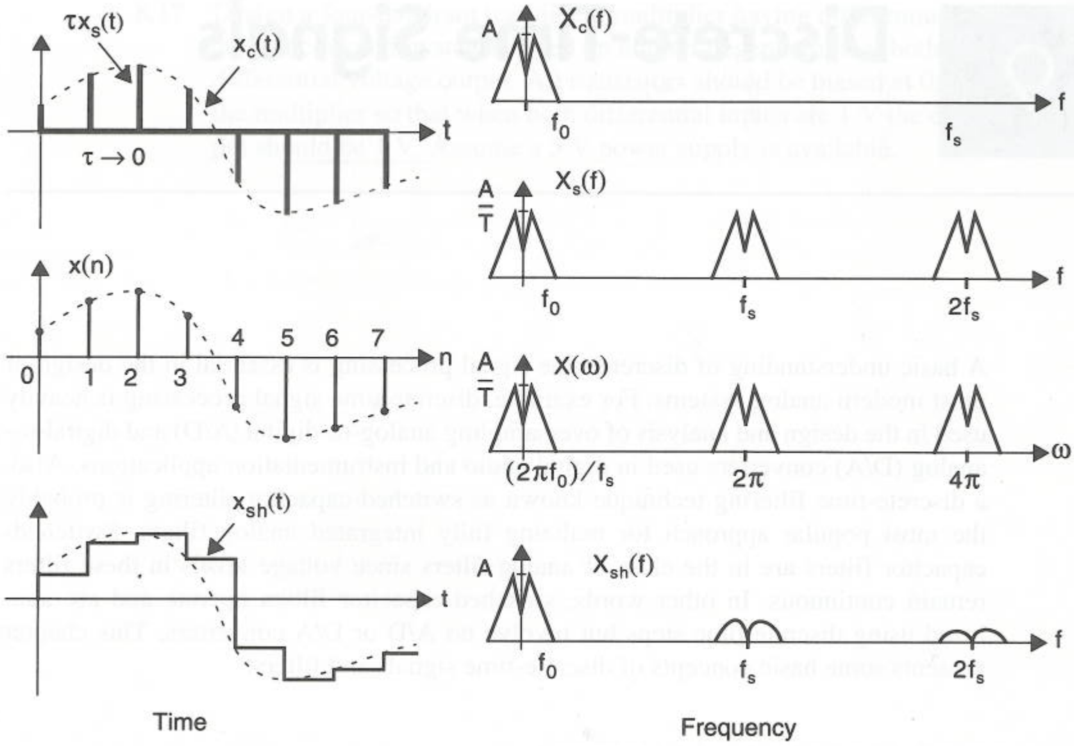

Fig. 9.2 Some time signals and frequency spectra.

example time signals and frequency spectra for this system are shown in Fig. 9.2. Here, $s(t)$ is a periodic impulse train in time with a period of T, where T equals the inverse of the sampling frequency, f_s. Some relationships for the signals in Figs. 9.1 and 9.2 are as follows:

1. $x_s(t)$ has the same frequency spectrum as $x_c(t)$, but the baseband spectrum repeats every f_s (assuming that no aliasing occurs and hence, an anti-aliasing filter is needed).
2. $x(n)$ has the same frequency spectrum as $x_s(t)$, but the sampling frequency is normalized to 1.
3. The frequency spectrum for $X_{sh}(t)$ equals that of $X_s(t)$ multiplied by a response of $(\sin x)/x$ (in effect, multiplying $X_s(f)$ by this response helps to remove high-frequency images).

The remainder of this chapter confirms these spectral relationships and introduces other basic discrete-time concepts.

9.2 LAPLACE TRANSFORMS OF DISCRETE-TIME SIGNALS

Consider the sampled signal, $x_s(t)$, related to the continuous-time signal, $x_c(t)$, as shown in Fig. 9.3. Here, $x_s(t)$ has been scaled by τ such that the area under the pulse

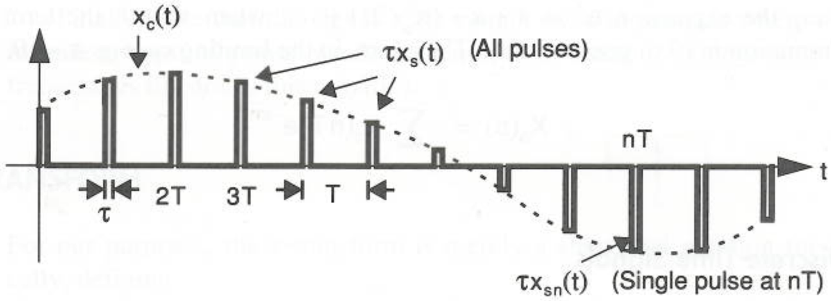

Fig. 9.3 Sampled and continuous-time signals.

at nT equals the value of $x_c(nT)$. In other words, at $t = nT$, we have

$$x_s(nT) = \frac{x_c(nT)}{\tau} \tag{9.1}$$

such that the area under the pulse, $\tau x_s(nT)$, equals $x_c(nT)$. Thus, as $\tau \to 0$, the height of $x_s(t)$ at time nT goes to ∞, and so we plot $\tau x_s(t)$ instead of $x_s(t)$.

We define $\vartheta(t)$ to be the step function given by

$$\vartheta(t) \equiv \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases} \tag{9.2}$$

Then $x_s(t)$ can be represented as a linear combination of a series of pulses, $x_{sn}(t)$, where $x_{sn}(t)$ is zero everywhere except for a single pulse at nT. The single-pulse signal, $x_{sn}(t)$, can be written as

$$x_{sn}(t) = \frac{x_c(nT)}{\tau}[\vartheta(t - nT) - \vartheta(t - nT - \tau)] \tag{9.3}$$

so that we can now write $x_s(t)$ as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_{sn}(t) \tag{9.4}$$

Note that these signals are defined for *all time,* so we can find the Laplace transform of $x_s(t)$ in terms of $x_c(t)$. Using the notation that $X_s(s)$ is the Laplace transform of $x_s(t)$, we find the Laplace transform $X_{sn}(s)$ for $x_{sn}(t)$ to be given by

$$X_{sn}(s) = \frac{1}{\tau}\left(\frac{1 - e^{-s\tau}}{s}\right)x_c(nT)e^{-snT} \tag{9.5}$$

Since $x_s(t)$ is merely a linear combination of $x_{sn}(t)$, we also have (for $\tau \neq 0$)

$$X_s(s) = \frac{1}{\tau}\left(\frac{1 - e^{-s\tau}}{s}\right)\sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT} \tag{9.6}$$

Using the expansion $e^x = 1 + x + (x^2/2!) + \cdots$, when $\tau \to 0$, the term before the summation in (9.6) goes to unity. Therefore, in the limiting case as $\tau \to 0$, we have

$$X_s(s) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-snT} \tag{9.7}$$

Spectra of Discrete-Time Signals

The spectrum of the sampled signal, $x_s(t)$, can be found by replacing s by $j\omega$ in (9.7). However, a more intuitive approach to find the spectrum of $x_s(t)$ is to recall that multiplication in the time domain is equivalent to convolution in the frequency domain. To use this fact, note that, for $\tau \to 0$, $x_s(t)$ can be written as the product

$$x_s(t) = x_c(t)s(t) \tag{9.8}$$

where $s(t)$ is a periodic pulse train, or mathematically,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{9.9}$$

where $\delta(t)$ is the unit impulse function, also called the *Dirac delta function*. It is well known that the Fourier transform of a periodic impulse train is another periodic impulse train. Specifically, the spectrum of $s(t)$, $S(j\omega)$, is given by

$$S(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) \tag{9.10}$$

Now writing (9.8) in the frequency domain, we have

$$X_s(j\omega) = \frac{1}{2\pi}X_c(j\omega) \otimes S(j\omega) \tag{9.11}$$

where $\otimes$ denotes convolution. Finally, by performing this convolution either mathematically or graphically, the spectrum of $X_s(j\omega)$ can be seen to be given by

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\omega - \frac{jk2\pi}{T}\right) \tag{9.12}$$

or, equivalently,

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j2\pi f - jk2\pi f_s) \tag{9.13}$$

Equations (9.12) and (9.13) show that the spectrum for the sampled signal, $x_s(t)$, equals a sum of shifted spectra of $x_c(t)$, and therefore no aliasing occurs if $X_c(j\omega)$ is band limited to $f_s/2$. The relation in (9.13) also confirms the example spectrum for $X_s(f)$, shown in Fig. 9.2. Note that, for a discrete-time signal, $X_s(f) = X_s(f \pm kf_s)$, where k is an arbitrary integer as seen by substitution in (9.13).

Finally, note that the signal $x_s(t)$ cannot exist in practice when $\tau \to 0$ since an infinite amount of power would be required to create it. (Integrating $X_s(f)$ over all frequencies illustrates this remark.)

9.3 z-TRANSFORM

For our purposes, the z-transform is merely a shorthand notation for (9.7). Specifically, defining

$$z \equiv e^{sT} \tag{9.14}$$

we can write

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x_c(nT) z^{-n} \tag{9.15}$$

where $X(z)$ is called the z-transform of the samples $x_c(nT)$.

Two properties of the z-transform that can be deduced from Laplace transform properties are as follows:

1. If $x(n) \leftrightarrow X(z)$, then $x(n-k) \leftrightarrow z^{-k}X(z)$
2. Convolution in the time domain is equivalent to multiplication in the frequency domain. Specifically, if $y(n) = h(n) \otimes x(n)$, where $\otimes$ denotes convolution, then $Y(z) = H(z)X(z)$. Similarly, multiplication in the time domain is equivalent to convolution in the frequency domain.

Note that $X(z)$ is not a function of the sampling rate but is related only to the numbers $x_c(nT)$, whereas $X_s(s)$ is the Laplace transform of the signal $x_s(t)$ as $\tau \to 0$. In other words, the signal $x(n)$ is simply a series of numbers that may (or may not) have been obtained by sampling a continuous-time signal. One way of thinking about this series of numbers as they relate to the samples of a possible continuous-time signal is that the original sample time, T, has been effectively normalized to one (i.e., $f'_s = 1$ Hz). Such a normalization of the sample time, T, in both time and frequency, justifies the spectral relation between $X_s(f)$ and $X(\omega)$ shown in Fig. 9.2. Specifically, the relationship between $X_s(f)$ and $X(\omega)$ is given by

$$X_s(f) = X\left(\frac{2\pi f}{f_s}\right) \tag{9.16}$$

or, equivalently, the following frequency scaling has been applied:

$$\omega = \frac{2\pi f}{f_s} \tag{9.17}$$

This normalization results in discrete-time signals having ω in units of radians/sample, whereas the original continuous-time signals have frequency units of cycles/second (hertz) or radians/second. For example, a continuous-time sinusoidal signal of 1 kHz when sampled at 4 kHz will change by $\pi/2$ radians between each sample.

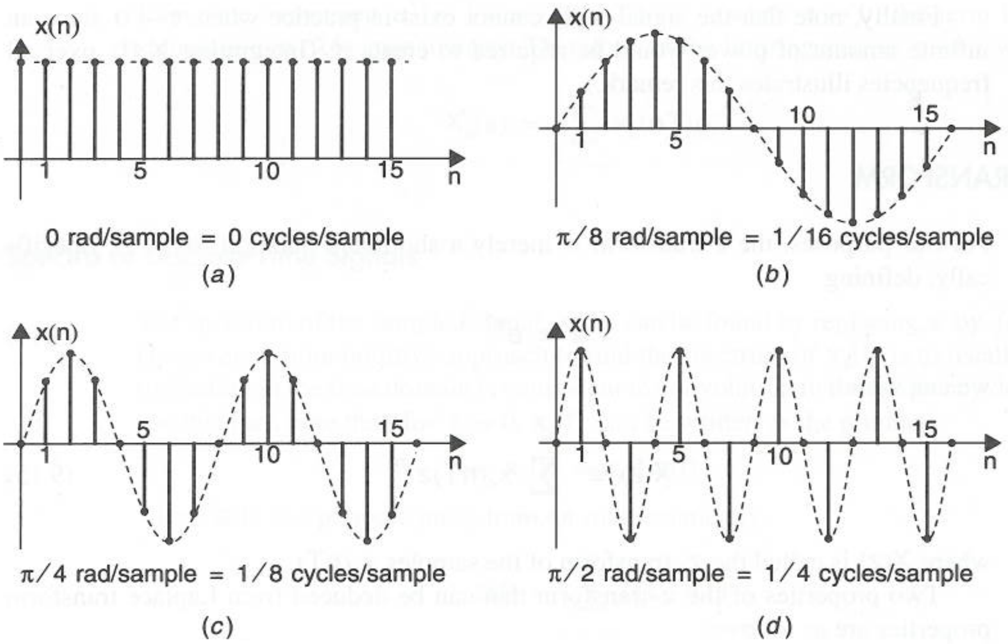

Fig. 9.4 Some discrete-time sinusoidal signals.

Therefore, such a discrete-time signal is defined to have a frequency of $\pi/2$ rad/sample. Other examples of discrete-time sinusoidal signals are shown in Fig. 9.4. It should be noted here that discrete-time signals are not unique since the addition of 2π results in the same signal. For example, a discrete-time signal that has a frequency of $\pi/4$ rad/sample is identical to that of $9\pi/4$ rad/sample. Thus, normally discrete-time signals are defined to have frequency components only between $-\pi$ and π rad/sample. For a more detailed discussion of this unit topic, see [Proakis, 1992].

EXAMPLE 9.1

Consider the spectra of $X_c(f)$ and $X_s(f)$, shown in Fig. 9.2, where f_0 is 1 Hz and f_s is 4 Hz. Compare the time and spectrum plots of $X_s(f)$ and $X_{s2}(f)$, where $X_{s2}(f)$ is sampled at 12 Hz. How does $X(\omega)$ differ between the two sampling rates?

Solution

By sampling at 12 Hz, the spectrum of $X_c(f)$ repeats every 12 Hz, resulting in the signals shown in Fig. 9.5.

Note that, for $X(\omega)$, 4 Hz is normalized to 2π rad/sample, whereas for $X_2(\omega)$, 12 Hz is normalized to 2π rad/sample.

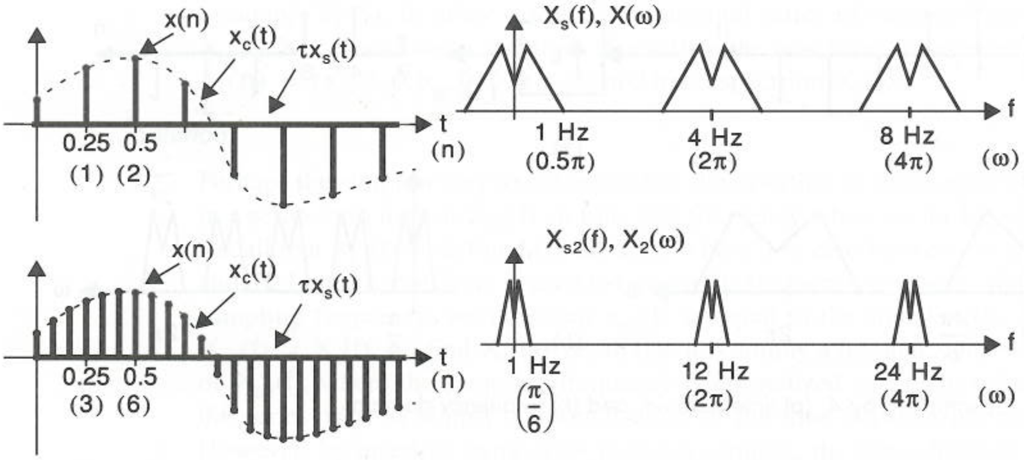

Fig. 9.5 Comparing time and frequency of two sampling rates.

9.4 DOWNSAMPLING AND UPSAMPLING

Two operations that are quite popular in discrete-time signal processing are downsampling and upsampling. Downsampling is used to reduce the sample rate (hopefully, without information loss), whereas upsampling is used to increase the sample rate. Although noninteger downsampling and upsampling rates can be achieved, here we consider only the case in which L is an integer value.

Downsampling is achieved by keeping every Lth sample and discarding the others. As Fig. 9.6 shows, the result of downsampling is to expand the original spectra by L. Thus, to avoid digital aliasing, the spectra of the original signal must be band limited to π/L before downsampling is done. In other words, the signal must be sampled L times above its minimum sampling rate so that no information is lost during downsampling.

Upsampling is accomplished by inserting $L-1$ zero values between samples, as shown in Fig. 9.7. In this case, one can show that the spectra of the resulting

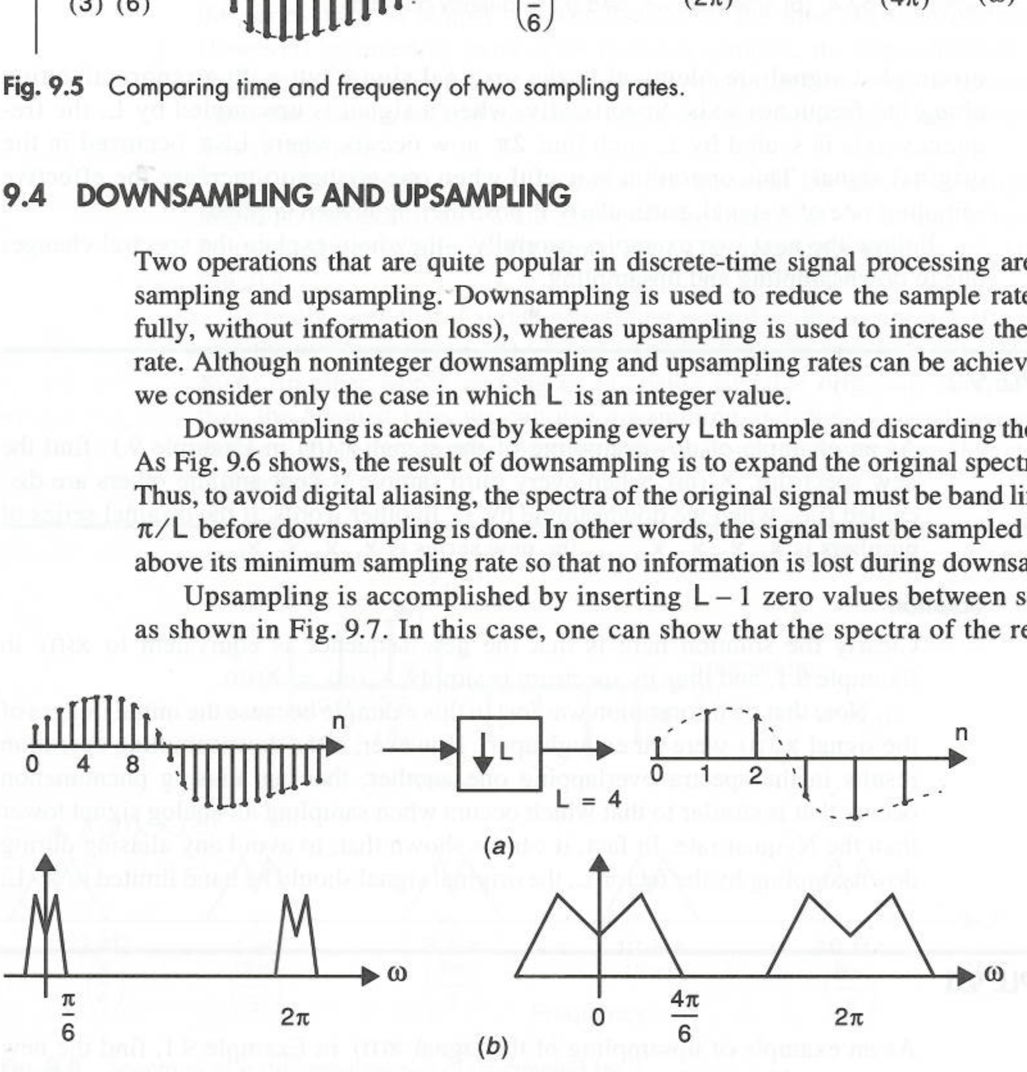

Fig. 9.6 Downsampling by 4: (a) time domain, and (b) frequency domain.

9.5 DISCRETE-TIME FILTERS

Thus far, we have seen the relationship between continuous-time and discrete-time signals in the time and frequency domain. However, often one wishes to perform filtering on a discrete-time signal to produce another discrete-time signal. In other words, an input series of numbers is applied to a discrete-time filter to create an output series of numbers. This filtering of discrete-time signals is most easily visualized with the shorthand notation of z-transforms.

Consider the system shown in Fig. 9.9, where the output signal is defined to be the impulse response, $h(n)$, when the input, $u(n)$, is an impulse (i.e., 1 for $n = 0$ and 0 otherwise). The transfer function of the filter is said to be given by $H(z)$, which is the z-transform of the impulse response, $h(n)$.

Frequency Response of Discrete-Time Filters

The transfer functions for discrete-time filters appear similar to those for continuous-time filters, except that, instead of polynomials in s, polynomials in z are obtained. For example, the transfer function of a low-pass, continuous-time filter, $H_c(s)$, might appear as

$$H_c(s) = \frac{4}{s^2 + 2s + 4} \tag{9.18}$$

The poles for this filter are determined by finding the roots of the denominator polynomial, which are $-1.0 \pm 1.7321j$ for this example. This continuous-time filter is also defined to have two zeros at ∞ since the denominator polynomial is two orders higher than the numerator polynomial. To find the frequency response of $H_c(s)$, the poles and zeros can be plotted in the s-plane (Fig. 9.10(*a*)), and the substitution $s = j\omega$ is equivalent to finding the magnitude and phase of vectors from a point along the $j\omega$ axis to all the poles and zeros.

An example of a discrete-time, low-pass transfer function is given by the following equation:

$$H(z) = \frac{0.05}{z^2 - 1.6z + 0.65} \tag{9.19}$$

Here, the poles now occur at $0.8 \pm 0.1j$ in the z-plane, and two zeros are again at ∞. To find the transfer function of $H(z)$, the poles and zeros can be plotted in the z-plane; however, instead of going along the vertical $j\omega$ axis as in the s-plane,

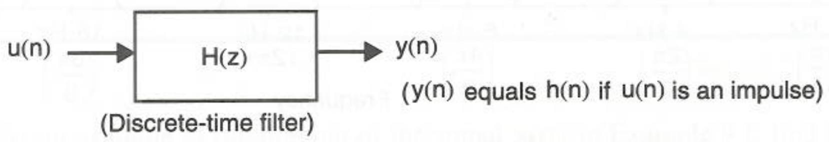

Fig. 9.9 Discrete-time filter system.

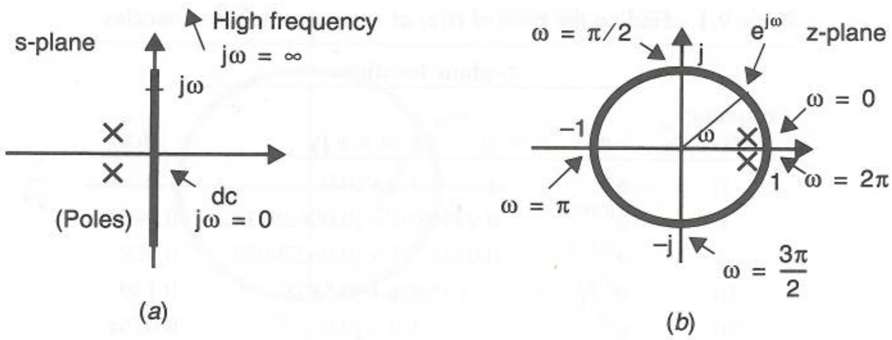

Fig. 9.10 Transfer function response. (*a*) Continuous-time, and (*b*) discrete-time.

the unit circle contour is used, so that $z = e^{j\omega}$, as shown in Fig. 9.10(*b*). Note that substituting $z = e^{j\omega}$ in the z-domain transfer function, $H(z)$, is simply a result of substituting $s = j\omega$ into (9.14), where T has been normalized to 1, as discussed in Section 9.3. Also note that poles or zeros occurring at $z = 0$ do not affect the magnitude response of $H(z)$ since a vector from the origin to the unit circle always has a length of unity. However, they would affect the phase response.

We see here that in the discrete-time domain, $z = 1$ corresponds to the frequency response at both dc (i.e., $\omega = 0$) and at $\omega = 2\pi$. Also, the time normalization of setting T to unity implies that $\omega = 2\pi$ is equivalent to the sampling-rate speed (i.e., $f = f_s$) for $X_s(f)$. In addition, note that the frequency response of a filter need be plotted only for $0 \le \omega \le \pi$ (i.e., $0 \le \omega \le f_s/2$) since, for filters with real coefficients, the poles and zeros always occur in complex-conjugate pairs (or on the real axis), so the magnitude response of the filter is equal to that for $\pi \le \omega \le 2\pi$ (the filter's phase is antisymmetric). Going around the circle again gives the same result as the first time, implying that the frequency response repeats every 2π.

Before we leave this section, a word of caution is in order. To simplify notation, the same variables, f and ω, are used in both the continuous-time and discrete-time domains in Fig. 9.10. However, these variables are not equal in the two domains, and care should be taken not to confuse values from the two domains. The continuous-time domain is used here for illustrative reasons only since the reader should already be quite familiar with transfer function analysis in the s-domain. In summary, the unit circle, $e^{j\omega}$, is used to determine the frequency response of a system that has its input and output as a series of numbers, whereas the $j\omega$-axis is used for a system that has continuous-time inputs and outputs. However, ω is different for the two domains.

EXAMPLE 9.4

Assuming a sample rate of $f_s = 100$ kHz, find the magnitude of the transfer function in (9.19) for 0 Hz, 100 Hz, 1 kHz, 10 kHz, 50 kHz, 90 kHz, and 100 kHz.

Table 9.1 Finding the gain of H(z) at some example frequencies

Frequency (kHz)	z-plane locations		
	$e^{j\omega}$	$z = x + jy$	$\lvert H(z)\rvert$
0	e^{j0}	$1.0 + j0.0$	1.0
0.1	$e^{j0.002\pi}$	$0.9999803 + j0.00628314$	0.9997
1	$e^{j0.02\pi}$	$0.9980267 + j0.06279052$	0.968
10	$e^{j0.2\pi}$	$0.809 + j0.5878$	0.149
50	$e^{j\pi}$	$-1.0 + j0.0$	0.0154
90	$e^{j1.8\pi}$	$0.809 - j0.5878$	0.149
100	$e^{j2\pi}$	$1.0 + j0.0$	1.0

Solution

To find the magnitude of H(z) at these frequencies, first we find their equivalent z-domain locations by normalizing f_s to 2π. Next, we find the gain of H(z) by putting these z-domain values into (9.19) and finding the magnitude of the resulting complex value. Table 9.1 summarizes the results.

Note that the gain of H(z) is the same at both 0 Hz and 100 kHz, as expected, since their z-plane locations are the same. Also, the gain is the same at both 10 kHz and 90 kHz since their z-plane locations are complex conjugates of each other. Finally, note that the minimum gain for this transfer function occurs at $z = -1$, or equivalently, $f_s/2 = 50$ kHz.

EXAMPLE 9.5

Consider a first-order H(z) having its zero at ∞ and its pole on the real axis, where $0 < a < 1$. Mathematically, the transfer function is represented by $H(z) = b/(z - a)$. Find the value of ω, where the magnitude of H(z) is 3 dB lower than its dc value. What is the 3-dB value of ω for a real pole at 0.8? What fraction of the sampling rate, f_s, does it correspond to?

Solution

Consider the pole-zero plot shown in Fig. 9.11, where the zero is not shown since it is at ∞. Since the zero is at ∞, we need be concerned only with the magnitude of the denominator of H(z) and when it becomes $\sqrt{2}$ larger than its dc value. The magnitude of the denominator of H(z) for $z = 1$ (i.e., dc) is shown as the vector I_1 in Fig. 9.11. This vector changes in size as z goes around the unit circle and is shown as I_2 when it becomes $\sqrt{2}$ larger than I_1. Thus we can write,

$$I_2 = e^{j\omega} - a \equiv \sqrt{2}(1 - a) \tag{9.20}$$

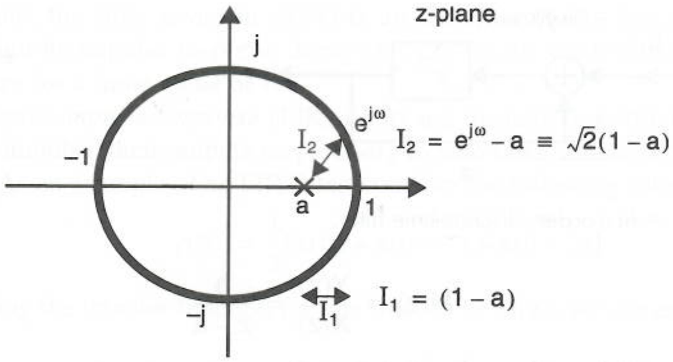

Fig. 9.11 Pole-zero plot used to determine the 3-dB frequency of a first-order, discrete-time filter.

Writing $e^{j\omega} = \cos(\omega) + j\sin(\omega)$, we have

$$|(\cos(\omega) - a) + j\sin(\omega)|^2 = 2(1-a)^2 \tag{9.21}$$

$$\cos^2(\omega) - (2a)\cos(\omega) + a^2 + \sin^2(\omega) = 2(1-a)^2 \tag{9.22}$$

$$1 - (2a)\cos(\omega) + a^2 = 2(1 - 2a + a^2) \tag{9.23}$$

which is rearranged to give the final result.

$$\omega = \cos^{-1}\left(2 - \frac{a}{2} - \frac{1}{2a}\right) \tag{9.24}$$

For $a = 0.8$, $\omega = 0.2241$ rad, or 12.84 degrees. Such a location on the unit circle corresponds to $0.2241/(2\pi)$ times f_s or, equivalently, $f_s/28.04$.

Stability of Discrete-Time Filters

To realize rational polynomials in z, discrete-time filters use delay elements (i.e., z^{-1} building blocks) much the same way that analog filters can be formed using integrators (i.e., s^{-1} building blocks). The result is that finite difference equations represent discrete-time filters rather than the differential equations used to describe continuous-time filters.

Consider the block diagram of a first-order, discrete-time filter shown in Fig. 9.12. A finite difference equation describing this block diagram can be written as

$$y(n+1) = bx(n) + ay(n) \tag{9.25}$$

In the z-domain, this equation is written as

$$zY(z) = bX(z) + aY(z) \tag{9.26}$$

where the z-domain property of delayed signals is used. We define the transfer function of this system to be $H(z)$ given by

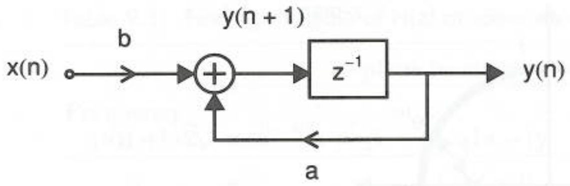

Fig. 9.12 A first-order, discrete-time filter.

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{b}{z-a} \tag{9.27}$$

which has a pole on the real axis at $z = a$.

To test for stability, we let the input, $x(n)$, be an impulse signal (i.e., 1 for $n = 0$ and 0 otherwise), which gives the following output signal, according to (9.25),

$$y(0) = k$$

where k is some arbitrary initial state value for y.

$$y(1) = b + ak$$

$$y(2) = ab + a^2k$$

$$y(3) = a^2b + a^3k$$

$$y(4) = a^3b + a^4k$$

$$\vdots$$

More concisely, the response, $h(n)$, is given by

$$h(n) = \begin{cases} 0 & (n<0) \\ k & (n=0) \\ (a^{n-1}b + a^n k) & (n \ge 1) \end{cases} \tag{9.28}$$

Clearly, this response remains bounded only when $|a| \le 1$ for this first-order filter and is unbounded otherwise.

Although this stability result is shown only for first-order systems, in general, an arbitrary, linear, time-invariant, discrete-time filter, $H(z)$, is stable if and only if all its poles are located within the unit circle. In other words, if z_{pi} are the poles, then $|z_{pi}| < 1$ for all i. Locating some poles on the unit circle is similar to poles being on the imaginary $j\omega$-axis for continuous-time systems. For example, in the preceding first-order example, if $a = 1$, the pole is at $z = 1$, and the system is marginally stable (in fact, it is a discrete-time integrator). If we let $a = -1$, this places the pole at $z = -1$, and one can show that the system oscillates at $f_s/2$, as expected.

IIR and FIR Filters

Infinite-impulse-response (IIR) filters are discrete-time filters that, when excited by an impulse, their outputs remain nonzero, assuming infinite precision arithmetic. For example, the filter given in (9.27) is an IIR filter (for a not equal to zero) since, although its impulse response decays to zero (as all stable filters should), it remains nonzero for a finite value of n.

Finite-impulse-response (FIR) filters are discrete-time filters that, when excited by an impulse, their outputs go precisely to zero (and remain zero) after a finite value of n. As an example of an FIR filter, consider the following filter,

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)] \tag{9.29}$$

Defining the transfer function for this filter to be $H(z)$, we can easily show that

$$H(z) = \frac{1}{3}\sum_{i=0}^{2} z^{-i} \tag{9.30}$$

This filter is essentially a running average filter since its output is equal to the average value of its input over the last three samples. Applying an impulse signal to this filter results in an output that is nonzero for only three samples and, therefore, this is an FIR type filter. Note that this FIR filter has poles, but they all occur at $z = 0$.

Some advantages of FIR filters are that stability is never an issue (they are always stable) and exact linear phase filters can be realized (a topic beyond the scope of this chapter). However, for many specifications, an IIR filter can meet the same specifications as an FIR filter, but with a much lower order, particularly in narrowband filters in which the poles of an IIR filter are placed close to the unit circle (i.e., has a slowly decaying impulse response).

Bilinear Transform

With modern filter design software, desired discrete-time transfer functions that meet specifications can be obtained entirely within the discrete-time domain. However, another approach draws on the wealth of knowledge of continuous-time transfer-function approximation and uses the bilinear transform. Assuming that $H_c(p)$ is a continuous-time transfer function (where p is the complex variable equal to $\sigma_p + j\Omega$), the bilinear transform is defined to be given by[1]

$$p = \frac{z-1}{z+1} \tag{9.31}$$

The inverse transformation is given by

$$z = \frac{1+p}{1-p} \tag{9.32}$$

1. It should be noted here that in many other textbooks, the bilinear transform is defined as $s = (2/T)[(z-1)/(z+1)]$ where T is the sampling period. Here, we have normalized T to 2 since we use the bilinear transform only as a temporary transformation to a continuous-time equivalent, and then we inverse transform the result back to discrete time. Thus, the value of T can be chosen arbitrarily as long as the same value is used in each transformation.

A couple of points of interest about this bilinear transform are that the z-plane locations of 1 and –1 (i.e., dc and $f_s/2$) are mapped to p-plane locations of 0 and ∞, respectively. However, with a little analysis, we will see that this bilinear transformation also maps the unit circle, $z = e^{j\omega}$, in the z-plane to the entire $j\Omega$-axis in the p-plane. To see this mapping, we substitute $z = e^{j\omega}$ into (9.31),

$$p = \frac{e^{j\omega}-1}{e^{j\omega}+1} = \frac{e^{j(\omega/2)}\left(e^{j(\omega/2)} - e^{-j(\omega/2)}\right)}{e^{j(\omega/2)}\left(e^{j(\omega/2)} + e^{-j(\omega/2)}\right)} \tag{9.33}$$

$$= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2)$$

Thus, we see that points on the unit circle in the z-plane are mapped to locations on the $j\Omega$-axis in the p-plane, and we have

$$\Omega = \tan(\omega/2) \tag{9.34}$$

As a check, note that the z-plane locations of 1 and –1, which correspond to ω equal to 0 and π, respectively, map to Ω equal to 0 and ∞.

One way to use this transform is to design a continuous-time transfer function, $H_c(p)$, and choose the discrete-time transfer function, $H(z)$, such that

$$H(z) \equiv H_c[(z-1)/(z+1)] \tag{9.35}$$

With such an arrangement, one can show that,

$$H(e^{j\omega}) = H_c[j\tan(\omega/2)] \tag{9.36}$$

and so the response of $H(z)$ is shown to be equal to the response of $H_c(p)$, except with a frequency warping according to (9.34). Note that the order of $H(z)$ equals that of $H_c(p)$ since, according to (9.35), each p term is replaced by another first-order function.

EXAMPLE 9.6

Using the bilinear transform, find the 3-dB frequency of a first-order discrete-time filter, $H(z)$, that has a pole at 0.8 and a zero at –1.

Solution

Using (9.31), we see that the z-plane pole of 0.8 and zero of –1 are mapped to a pole at –0.11111 and to a zero at ∞ in the p-plane, respectively. Such a continuous-time filter has a 3-dB frequency at Ω = 0.11111 rad/s. Using (9.34), we find that the equivalent 3-dB frequency value in the z-plane is given by ω = 0.2213 rad/sample, or $f_s/28.4$.

Note that this result is very close to that in Example 9.5. However, the two results are not in exact agreement since a zero at –1 is also present in this example.

EXAMPLE 9.7

Using the bilinear transform, find a first-order $H(z)$ that has a 3-dB frequency at $f_s/20$, a zero at -1, and a dc gain of 1.

Solution

Using (9.34), the frequency value, $f_s/20$, or equivalently, $\omega = (2\pi)/20 = 0.314159$ is mapped to $\Omega = 0.1584$ rad/s. Thus, $H_c(p)$ should have a 3-dB frequency value of 0.1584 rad/s. Such a 3-dB frequency value is obtained by having a p-plane zero equal to ∞ and a pole equal to -0.1584. Transforming these continuous-time pole and zero back into the z-plane using (9.32) results in a z-plane zero at -1 and a pole at 0.7265. Therefore, $H(z)$ appears as

$$H(z) = \frac{k(z+1)}{z-0.7265}$$

The constant k can be determined by setting the dc gain to 1, or equivalently, $|H(1)| = 1$, which results in $k = 0.1368$.

9.6 SAMPLE-AND-HOLD RESPONSE

In this section, we look at the frequency response that occurs when we change a discrete-time signal back into an analog signal with the use of a sample-and-hold circuit. Note that here we plot a frequency response for all frequencies (as opposed to only up to $f_s/2$) since the output signal is a continuous-time signal rather than a discrete-time one.

A sample-and-hold signal, $x_{sh}(t)$, is related to its sampled signal by the mathematical relationship

$$x_{sh}(t) = \sum_{n=-\infty}^{\infty} x_c(nT)[\vartheta(t-nT) - \vartheta(t-nT-T)] \tag{9.37}$$

Note that, once again, $x_{sh}(t)$ is well defined for all time, and thus the Laplace transform can be found to be equal to

$$\begin{aligned} X_{sh}(s) &= \frac{1-e^{-sT}}{s} \sum_{n=-\infty}^{\infty} x_c(nT) e^{-snT} \\ &= \frac{1-e^{sT}}{s} X_s(s) \end{aligned} \tag{9.38}$$

This result implies that the hold transfer function, $H_{sh}(s)$, is equal to

$$H_{sh}(s) = \frac{1-e^{-sT}}{s} \tag{9.39}$$