Sampling and spectra

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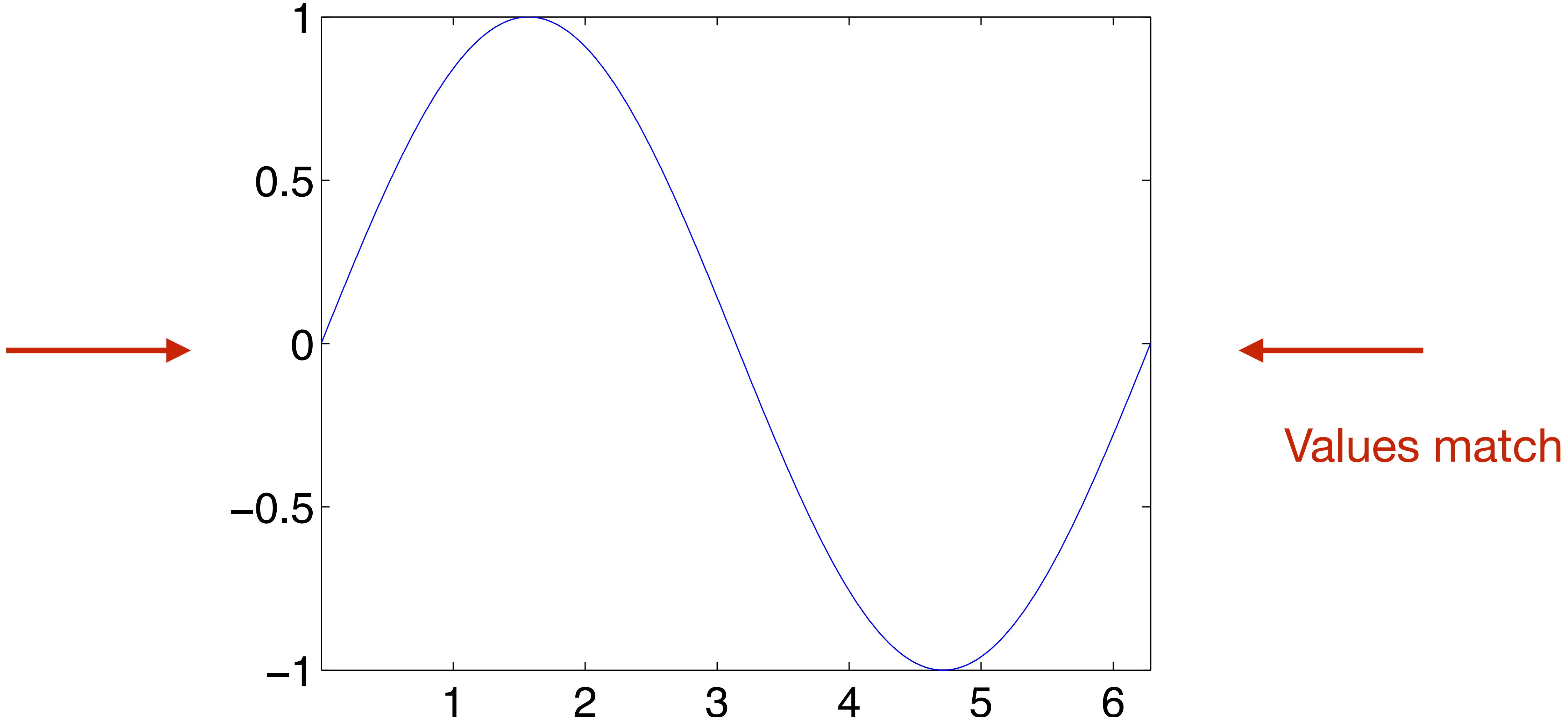
Goal

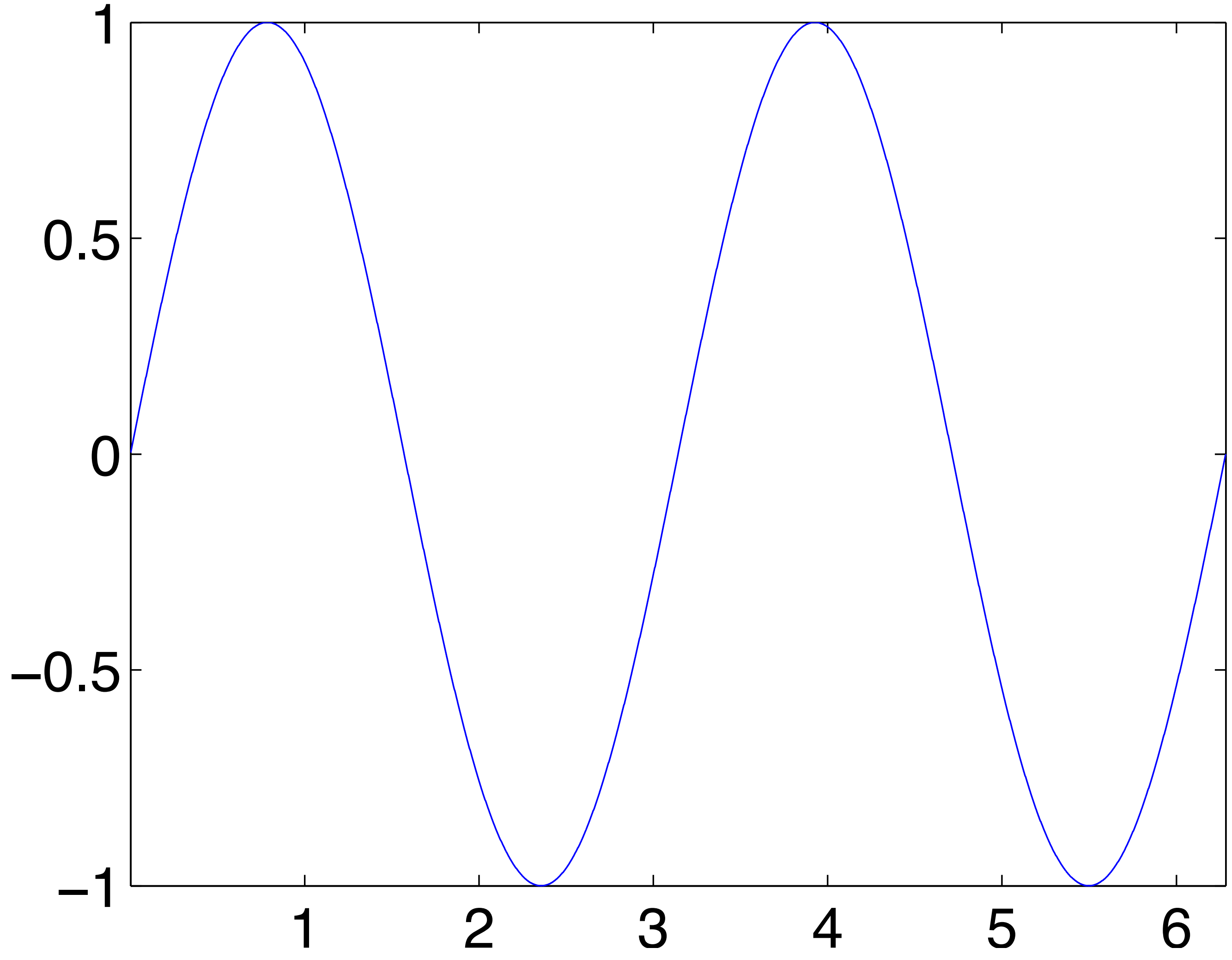
- Refresh some fundamental signal relations
  - Not intended as definitive treatment!
- Investigate discretization of signals in time ("sampling")
- Refer to Maloberti, Chapter 1
  - See reading directions

Periodic signal



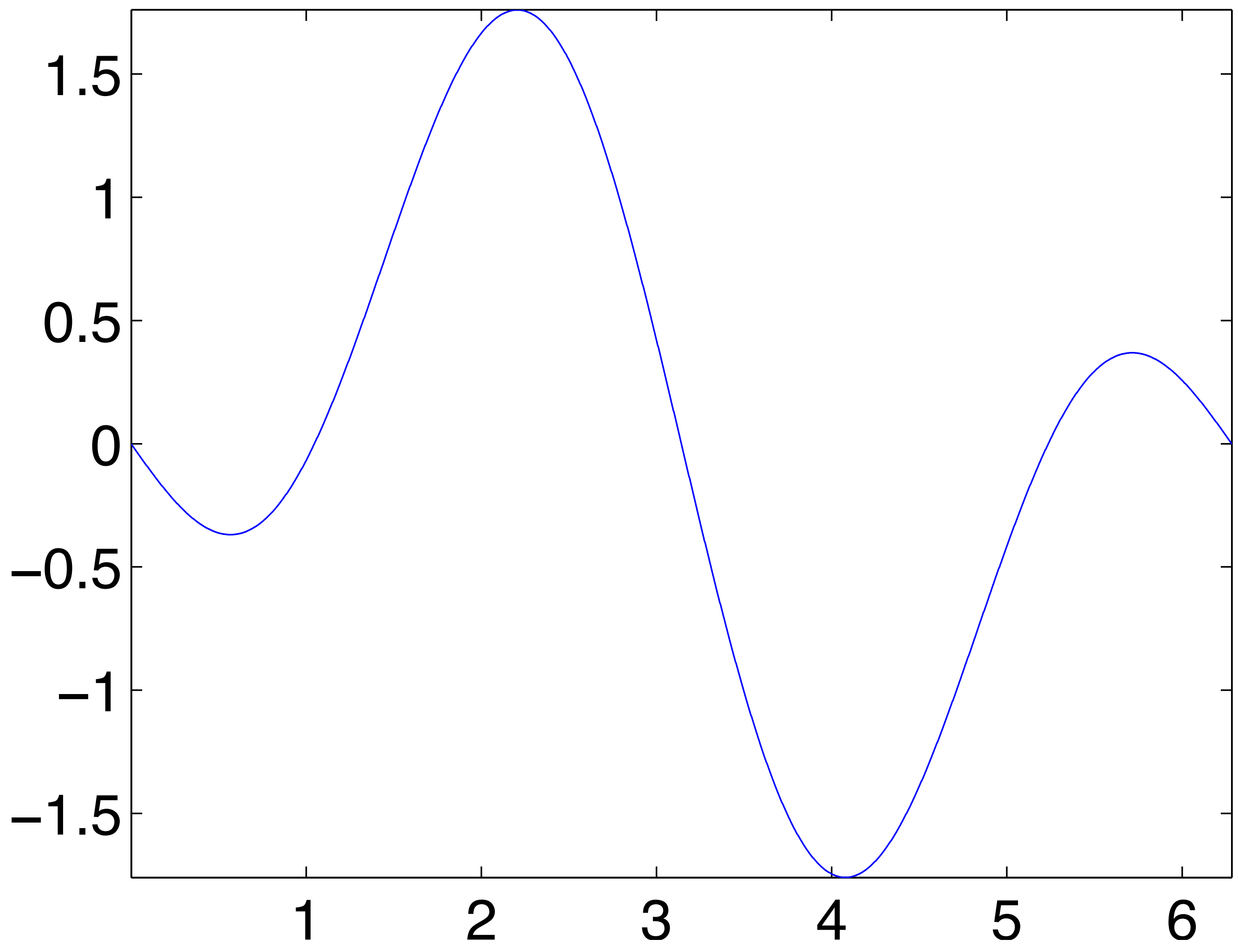
$$y = sin(x)$$

Periodic signal



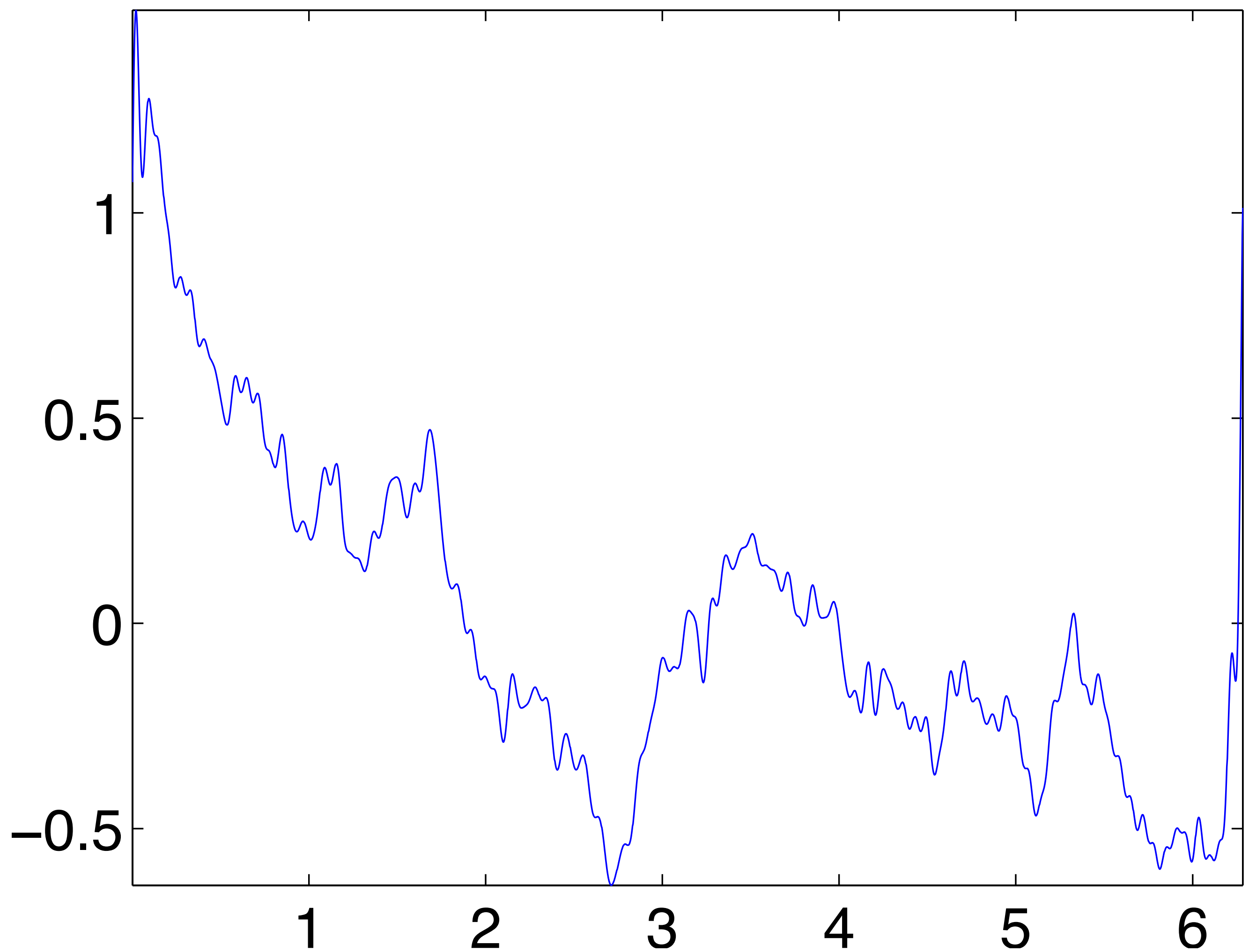
$$y = sin(2x)$$

Periodic signal



$$y = sin(x) - sin(2x)$$

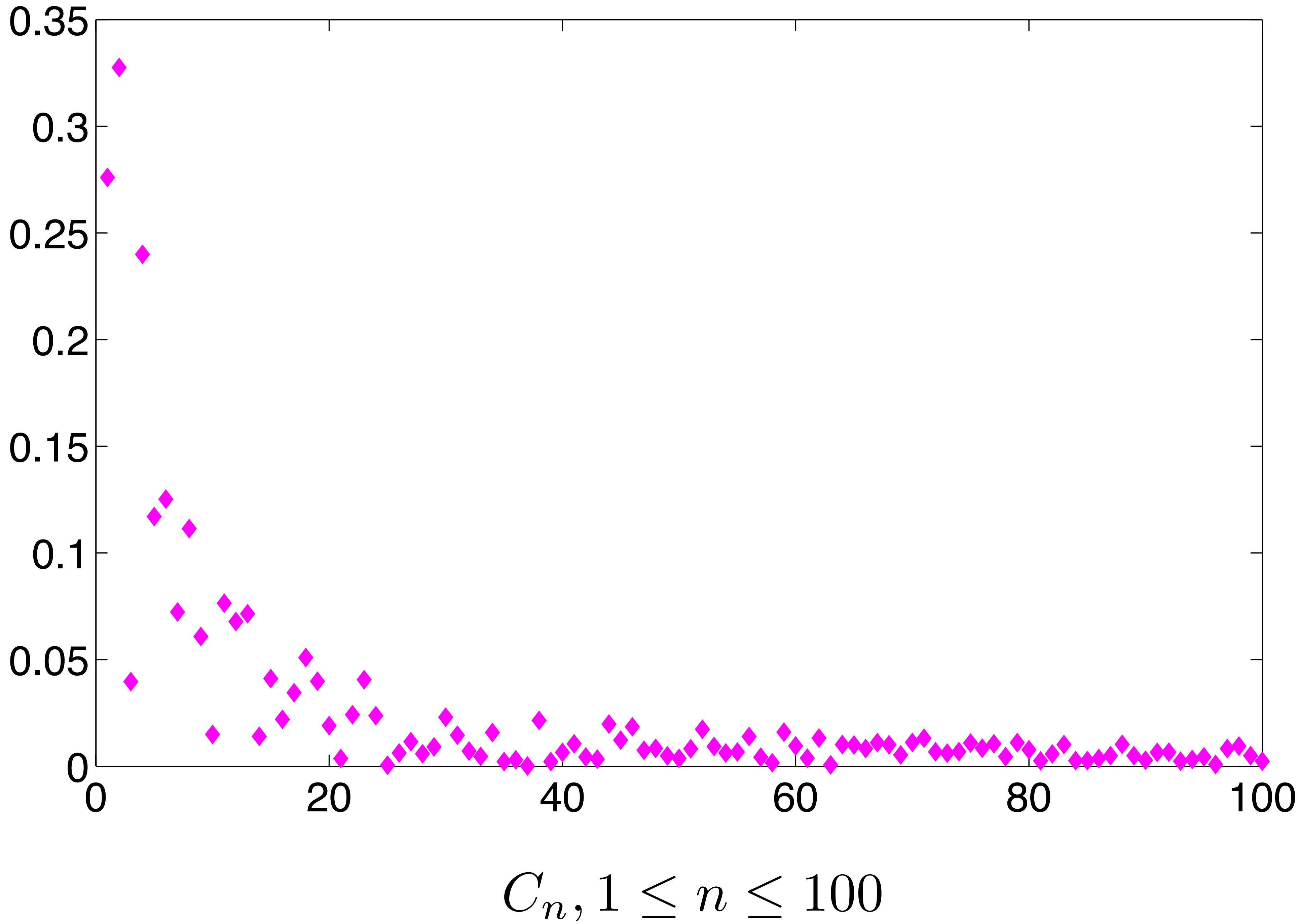
Periodic signal



$$y = \sum_{n=1}^{100} C_n sin(nx + \phi_n)$$

random coeffs

C_n coefficients



Fourier series

- Any 2π-periodic function f(x) can be expressed as sum of sines/cosines:

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}(a_n cos(nx) + b_n sin(nx))$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cos(nx)dx$$

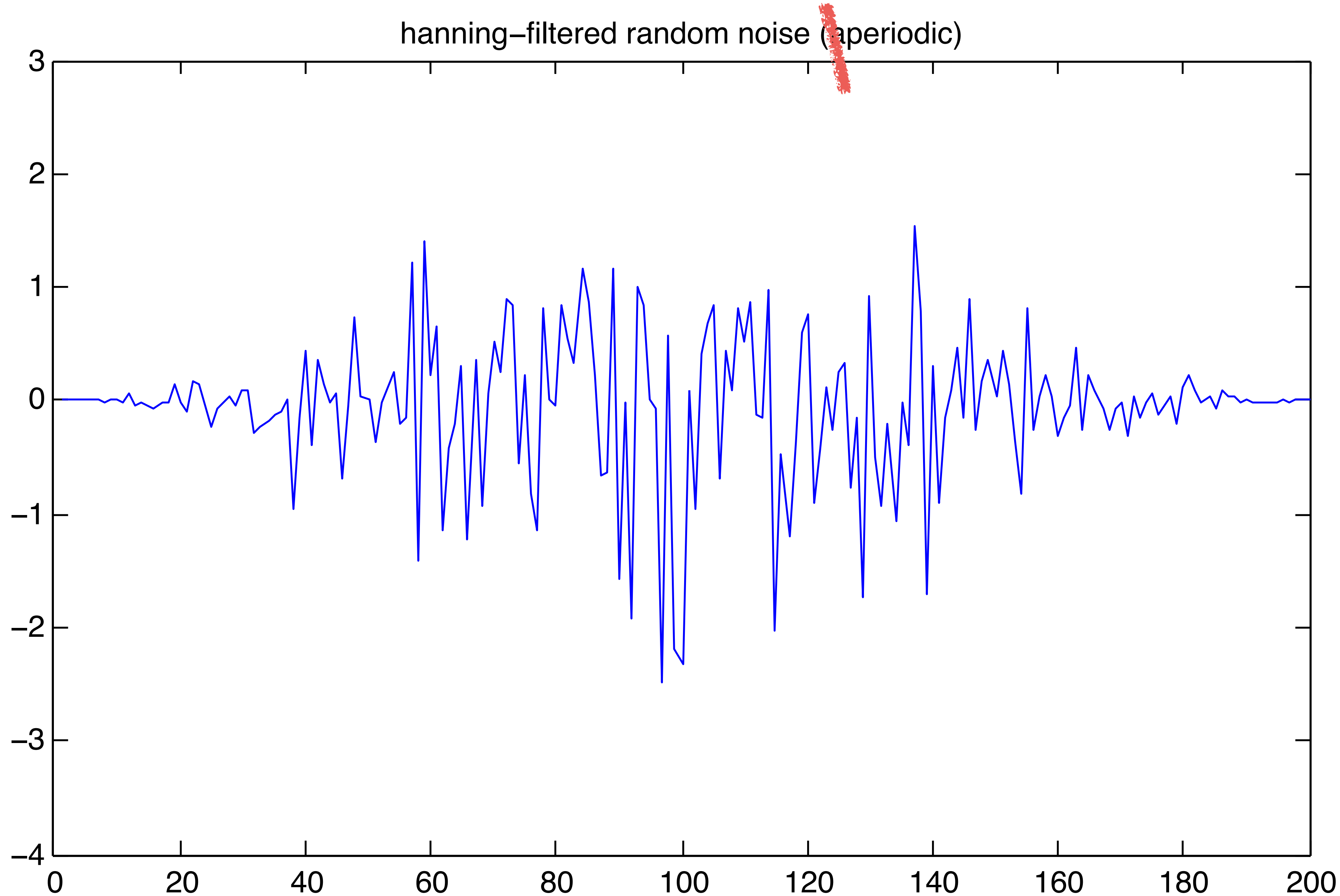
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(nx)dx$$

$$a_n cos(nx) + b_n sin(nx) = C_n sin(nx + \phi_n)$$

Spectrum

- Describes a signal in the frequency domain
  - One-to-one mapping from time domain
- We often use “power spectrum”: square amplitudes, ignore phases
  - Many-to-one mapping
- Esp. useful when studying Linear and Time-Invariant (LTI) systems
  - …which approximate many practical systems

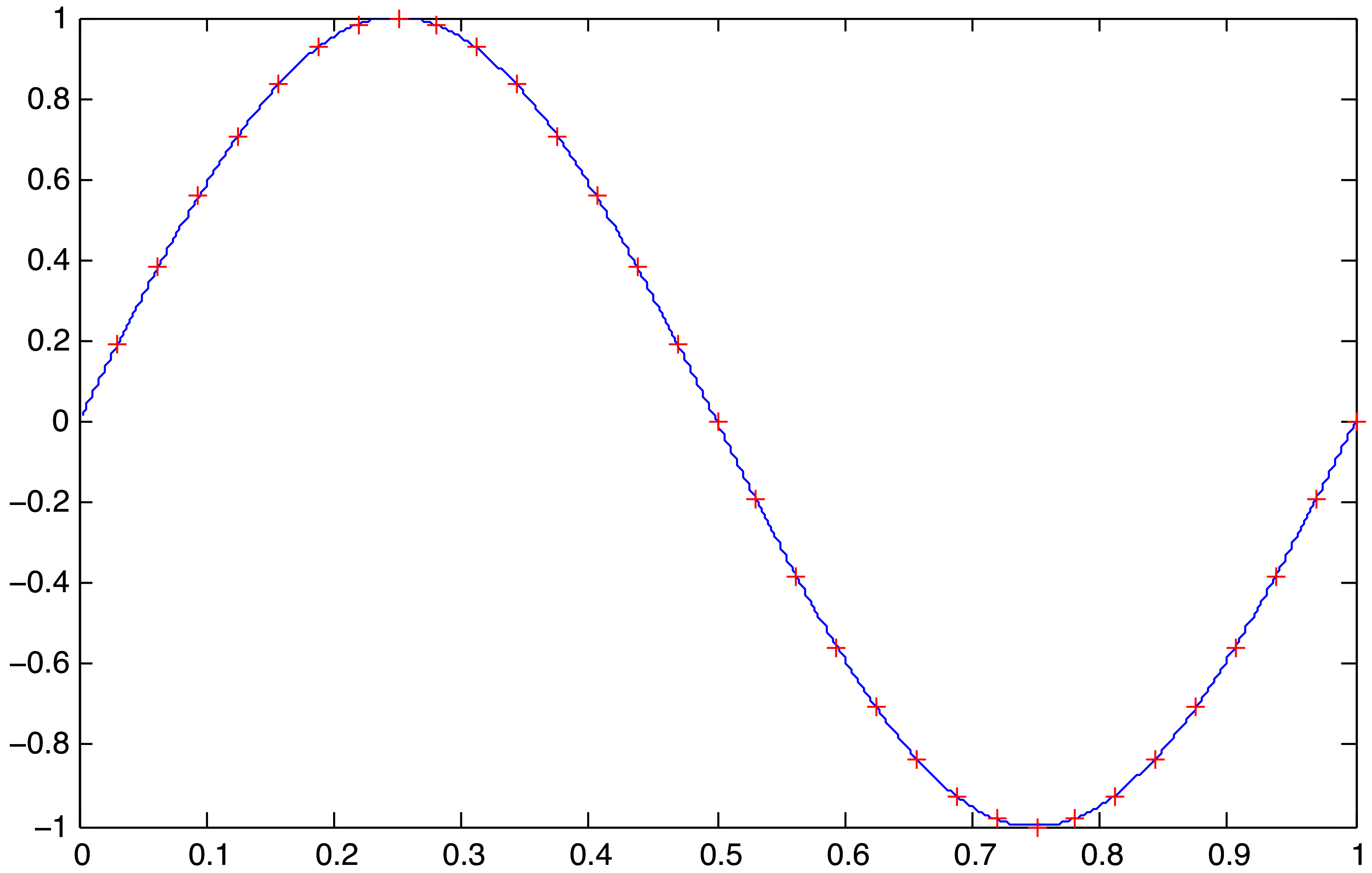
Aperiodic signals?



- Multiply with <u>window function</u>

Hanning window

Sampling



$f_s = 32$

$y = \sin(2\pi \cdot t), 0 < t \leq 1$

$y = \sin(2\pi \cdot t_k), t_k = (k / f_s), k = 0,1,\ldots$

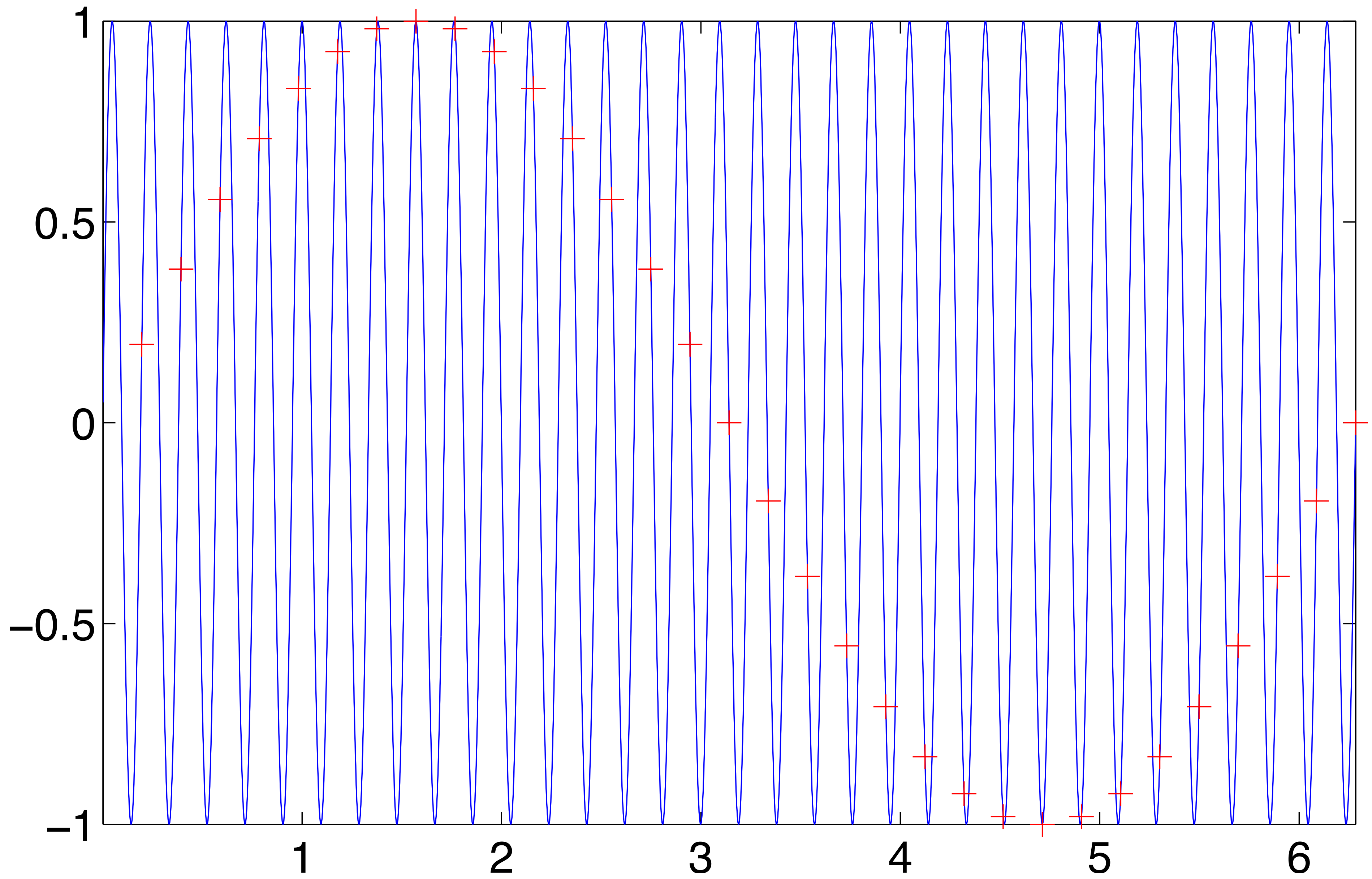
Nyquist-Shannon sampling theorem

Approximate

- ~~Exact~~ reconstruction of continuous signal from equidistant samples is possible if ~~no~~ small spectral components with $f \geq f_s / 2$
- Nyquist sampling frequency

Practical case…

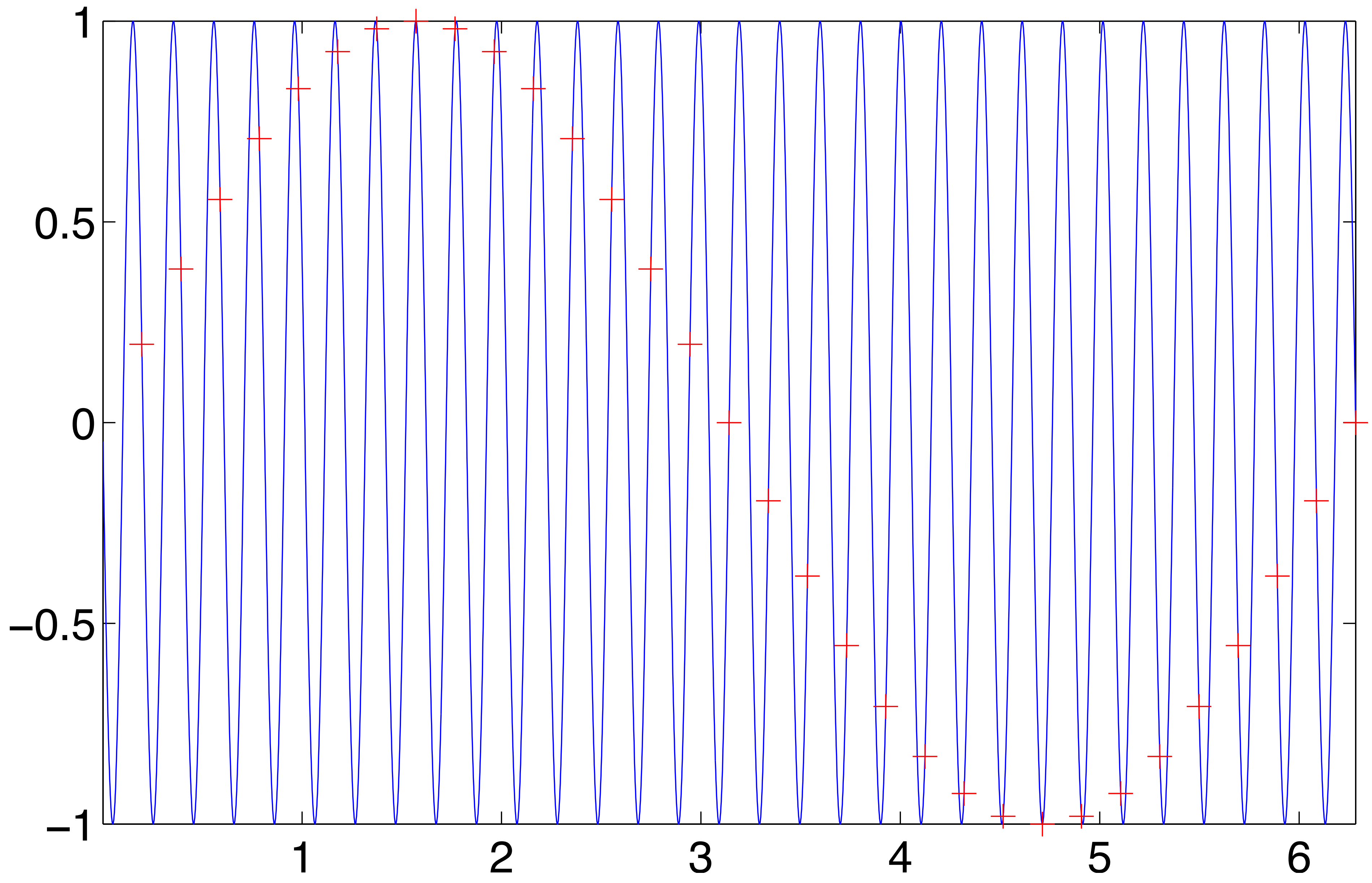
Nyquist violation



$$y = sin(33x), 0 < x \leq 2\pi$$

$$f_{sig} > f_s/2$$

Nyquist violation



$$y = -sin(31x), 0 < x \leq 2\pi$$

Aliases

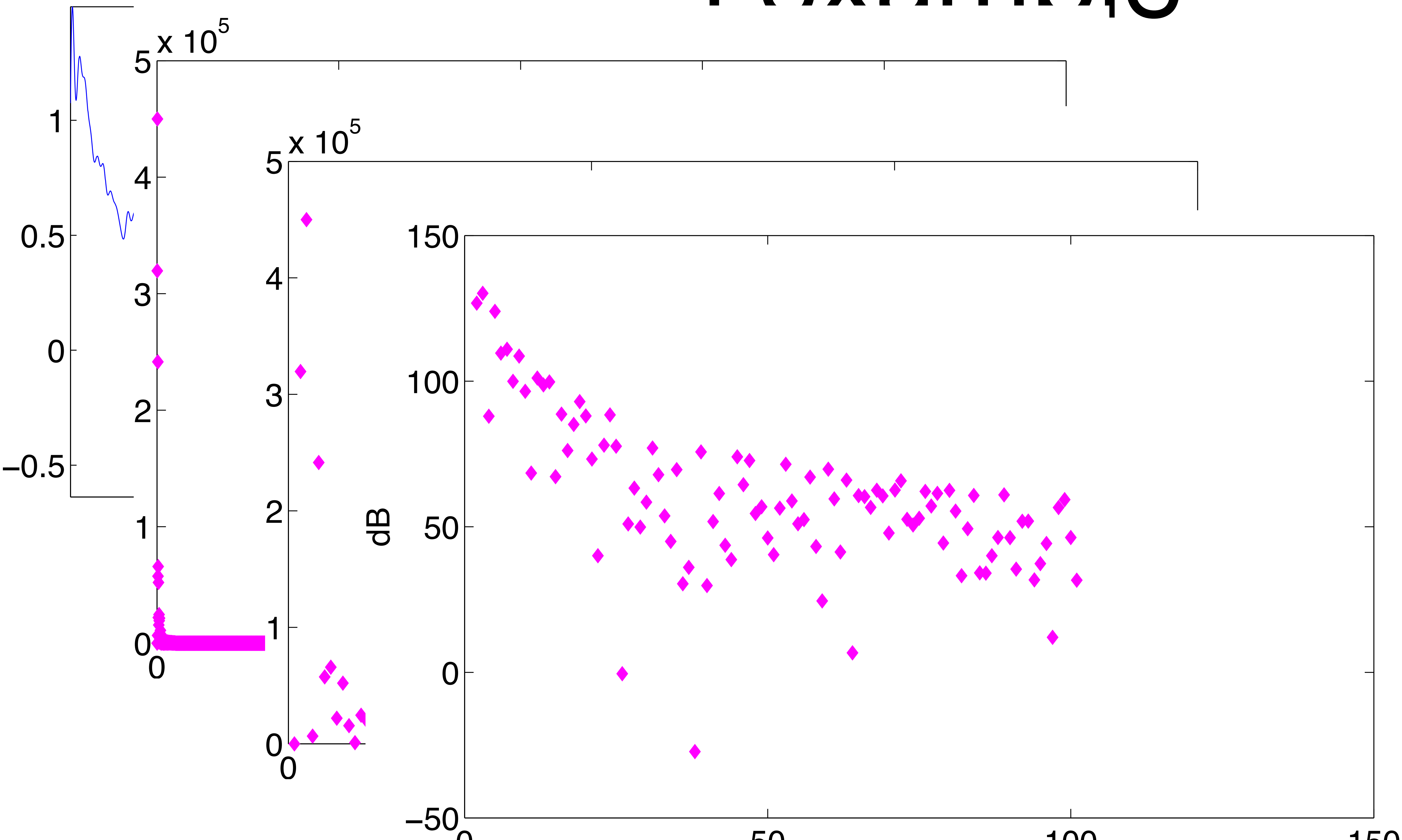
- Infinite number of continuous-time signals coincide with the same sampled-time version!
- Nyquist criterion selects one of these
- Note: possible (and sometimes useful) to select others
  - 2nd, 3rd etc. “Nyquist band”

Sampled-domain spectrum

$$S_k = \sum_{n=0}^{N-1} s_n \cdot e^{-i2\pi \frac{k}{N} n}$$

- Discrete-time Fourier transform (DFT)
  - FFT is a (class of) implementation(s) of the DFT
- Same number of components (N) as signal has samples
  - S_k is complex-valued
- We are often interested in power spectrum, $|S_k|^2$
  - $|S_k|^2$ is real-valued, ≥ 0, all k
  - Symmetric when s is real; consider lower half

Spectrum example

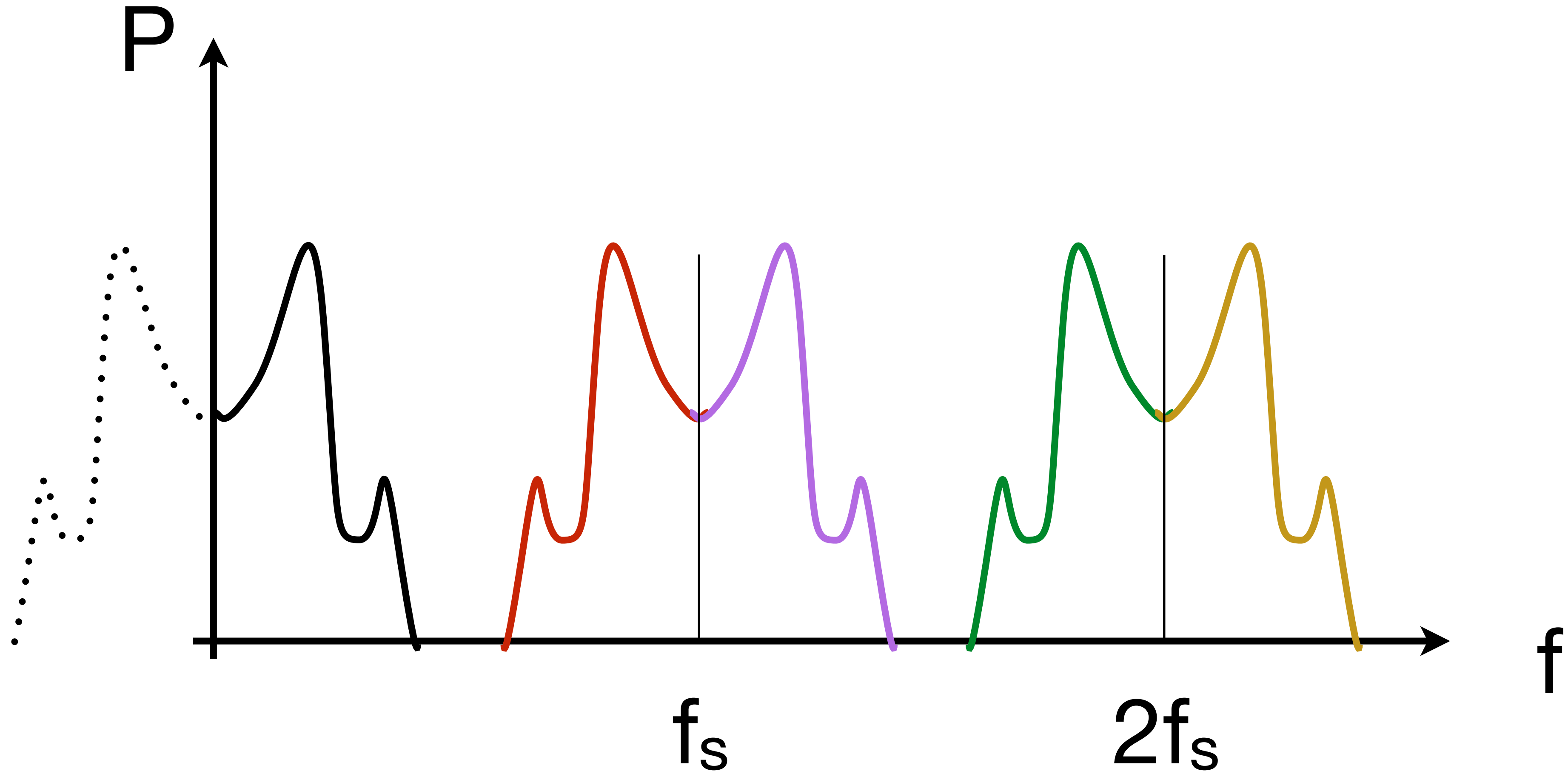


Sampling inaccuracies

1. Spectrum aliasing
2. Aperture window

3. Non-uniform sampling (next lecture)

1. Spectrum aliasing

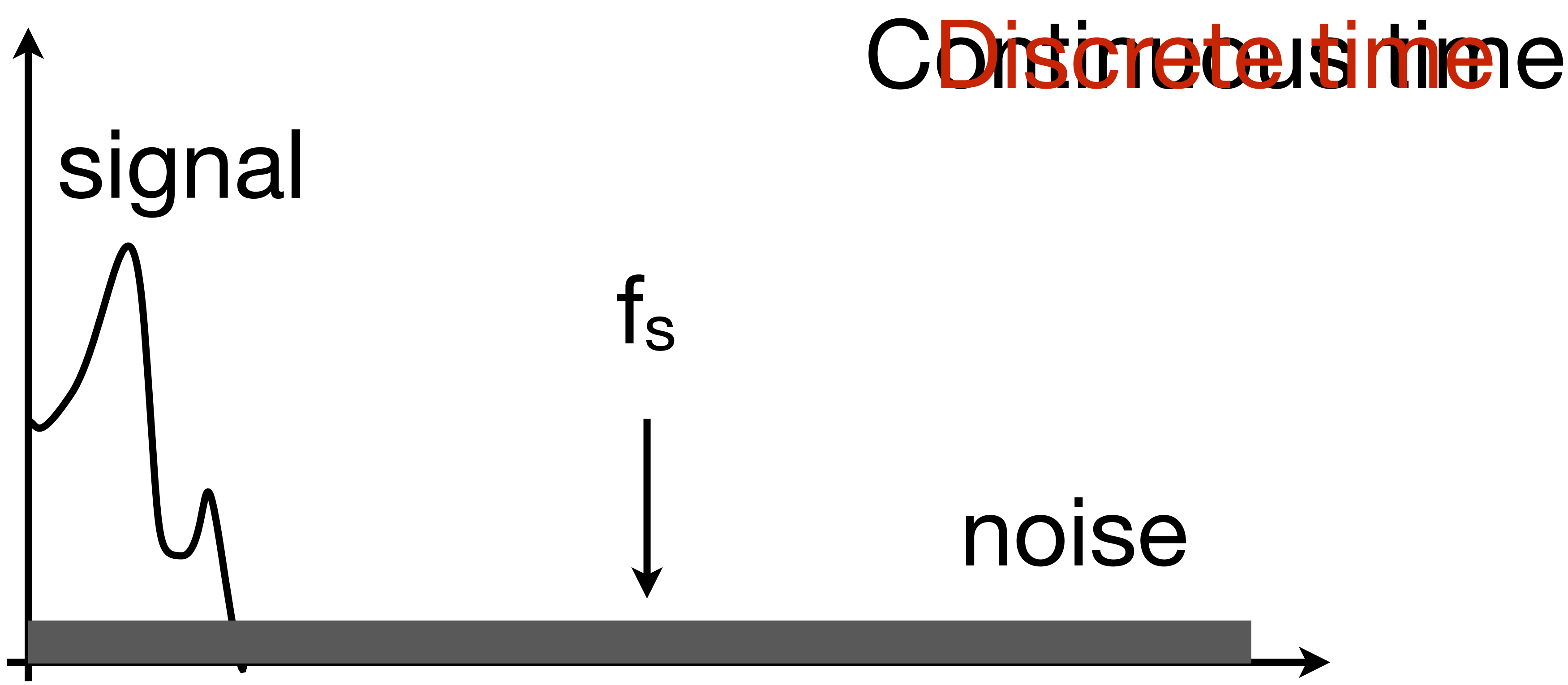


- Any discrete-time signal can correspond to many continuous-time signals
    - Indistinguishable in sampled domain!
- CT spectra “mirrored” around multiples of f_s

How get 1-1 mapping?

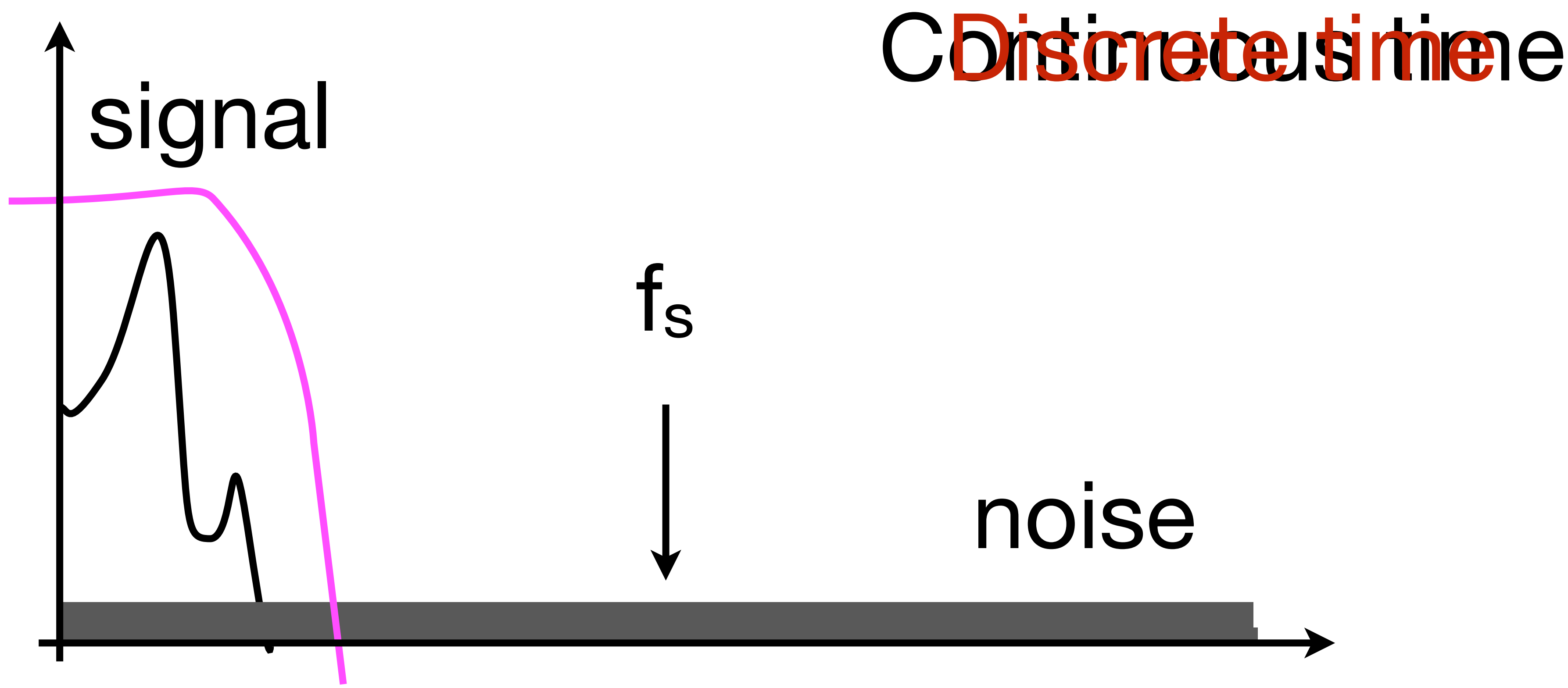
- Ensure CT signal contains only one of “equivalent” mirror images
- Most often, a low-pass filter is used
  - CT to ST: anti-aliasing filter
  - ST to CT: reconstruction filter

Noise folding



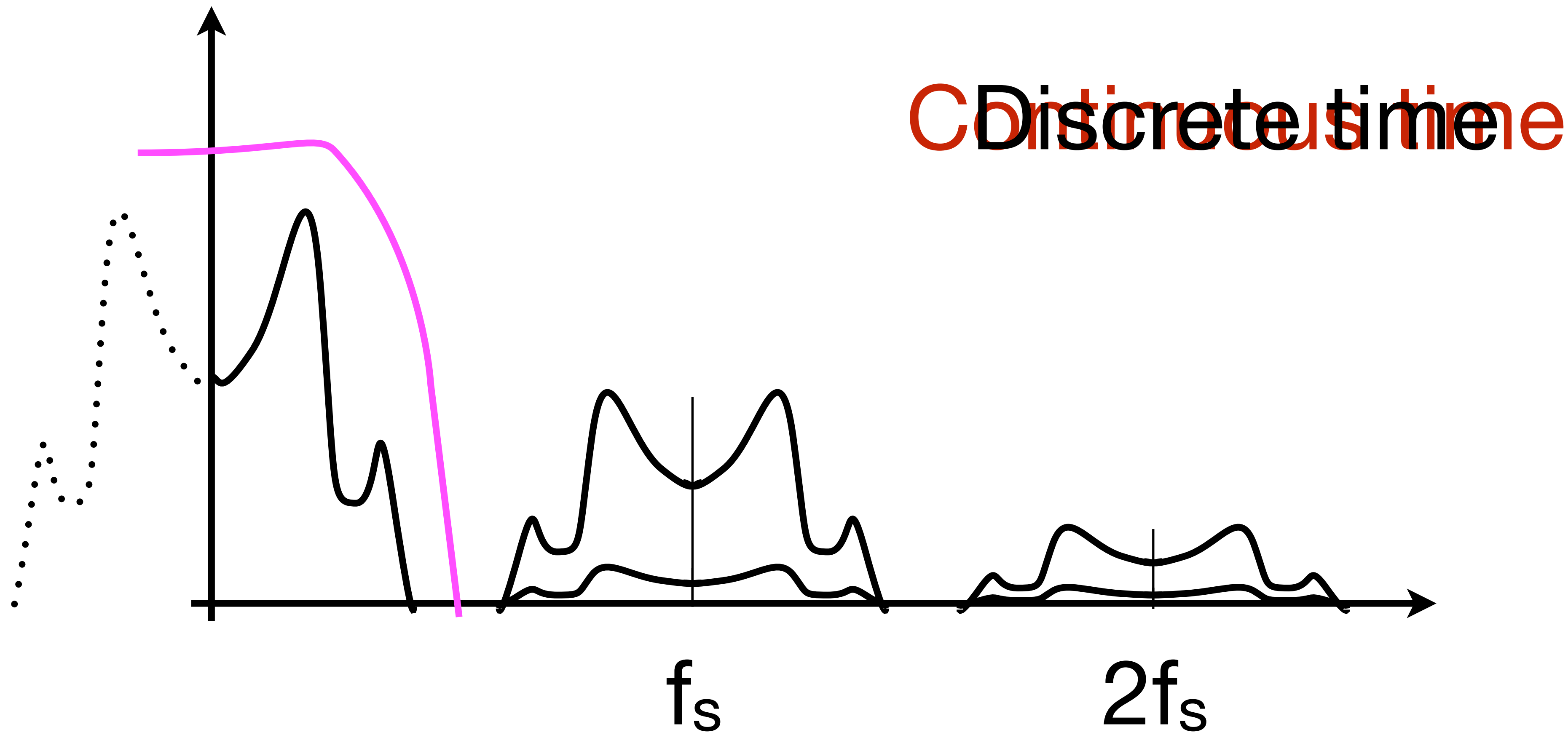
- Undesired signals (“noise”) corrupt desired signal
  - Added to the desired signal
- Once sampled, out-of-band noise is indistinguishable from desired signal!

Anti-alias filter



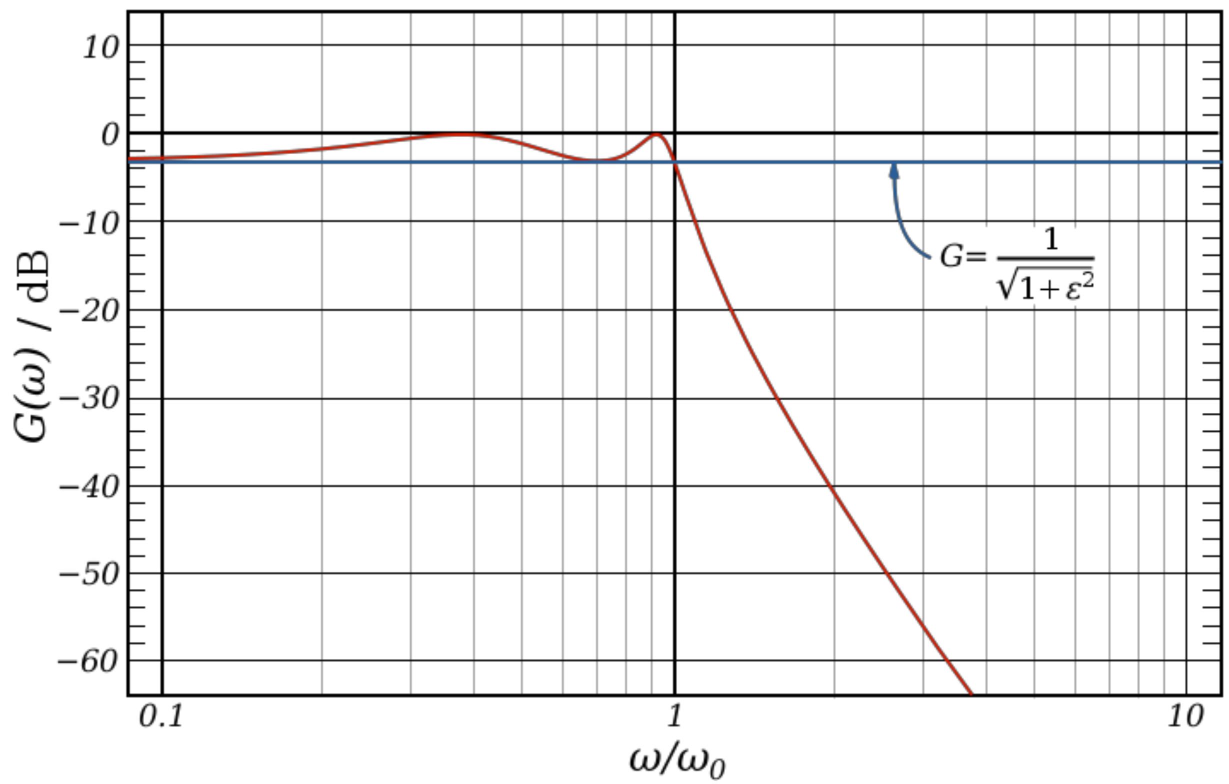
- Use frequency-selective filter to suppress out-of-band noise, before sampling and thus before aliasing
- Low-pass filter (here) selects first mirror image; band-pass for other image

Reconstruction filters

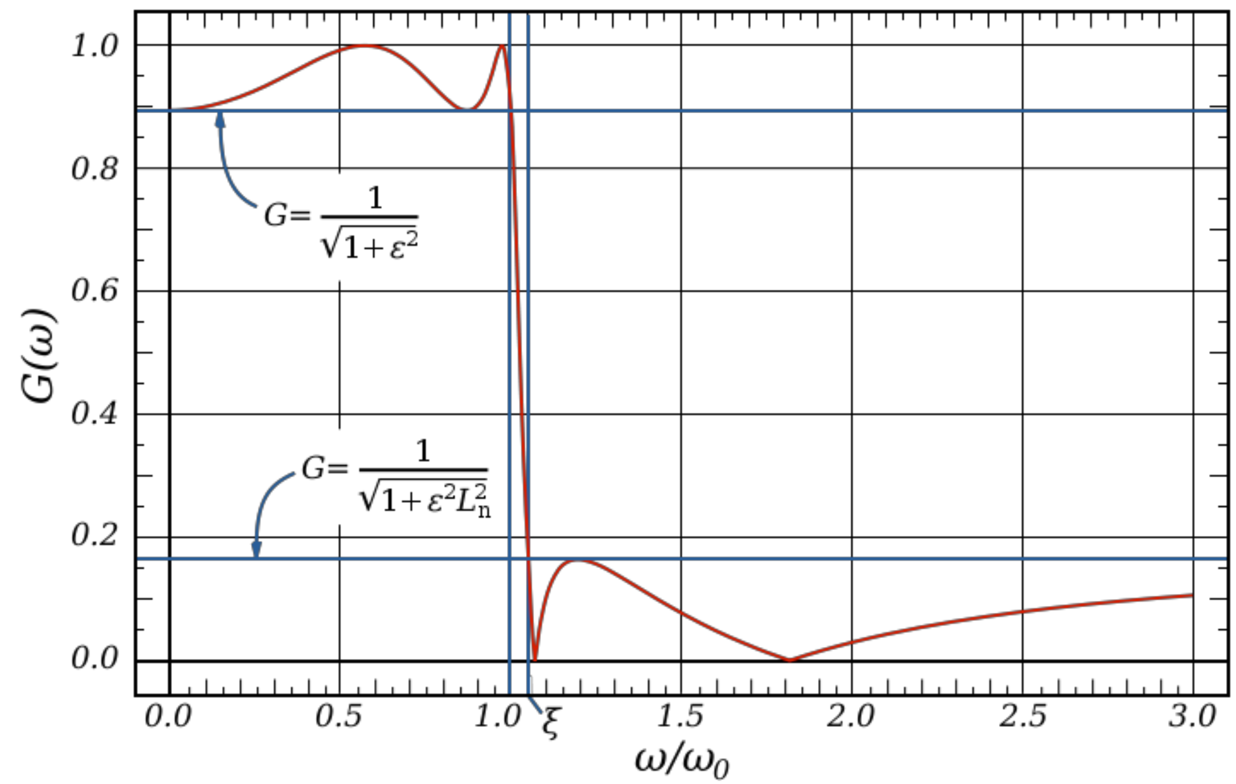


- Images in CT (general rolloff with frequency)
  - Rolloff rate depends on conversion details
- Again, filter selects one image

Continuous-time filters



4th order Chebyshev



4th order Cauer

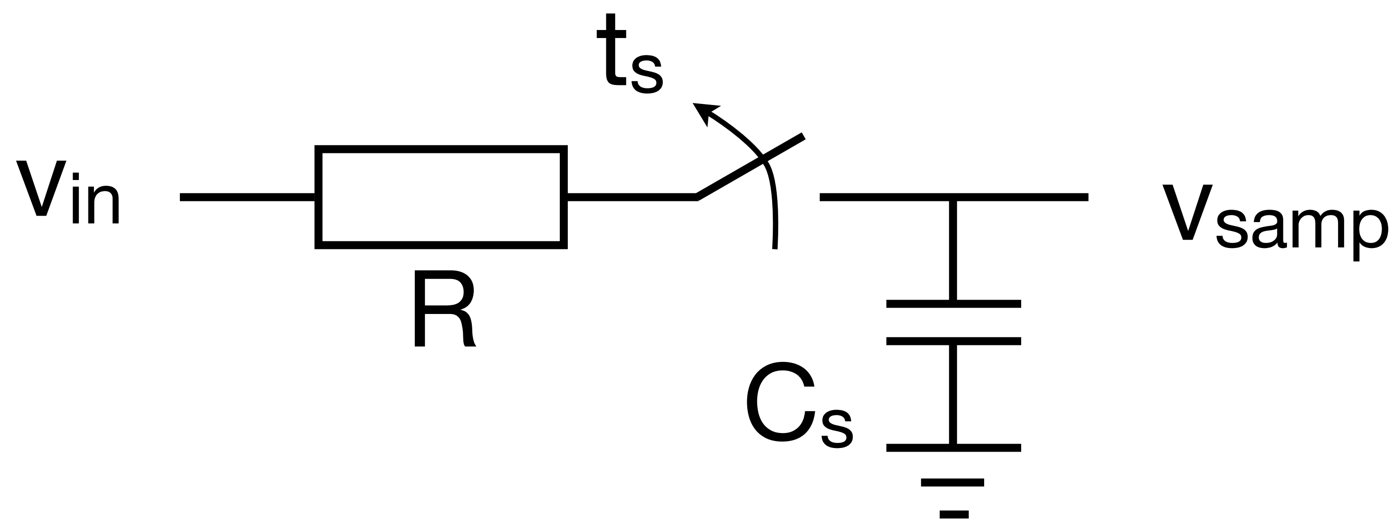
- Choices in pole/zero placement; examples…
- Suppression <u>never complete</u> in linear filters of limited order!
    - Increased suppression at a cost ($, W)

More in theme 4

[Wikipedia]

2. Aperture window

- Consider simple sample / hold circuit:



- Output follows input (filtered by RC) …

$$v_{samp}(t) = \int \alpha(\tau) v_{in}(t-\tau) d\tau$$

- … until switch opens at t_s

kernel

Sampling, ideal and not

- With Dirac kernel, sampling is ideal:

$$v_{samp}(t) = \int \delta(\tau) v_{in}(t-\tau) d\tau = v_{in}(t)$$

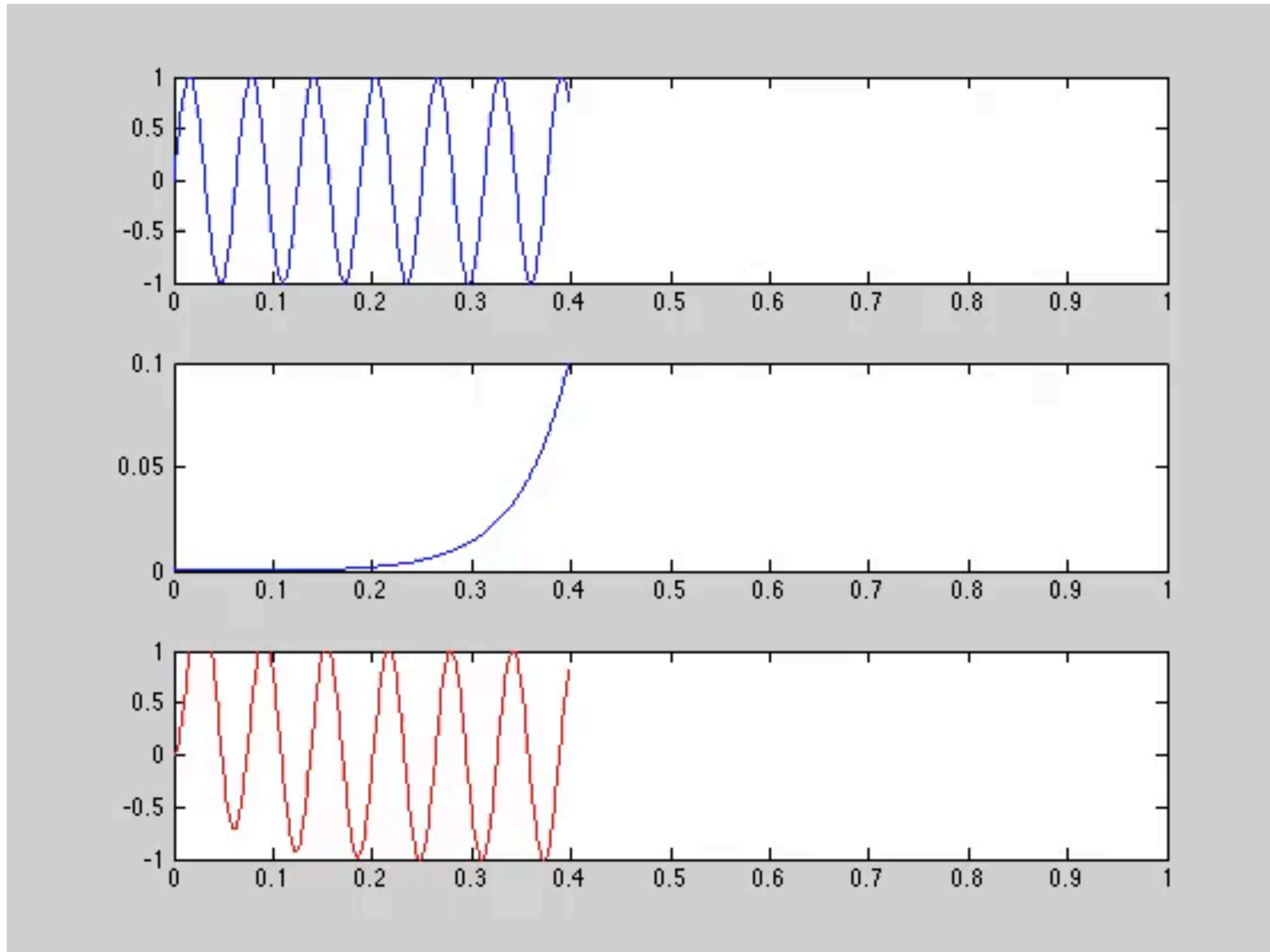
- O/w, convolution of input signal with a window function
    - "Aperture window"
    - In example S+H, window is an RC decay

Convolution

in

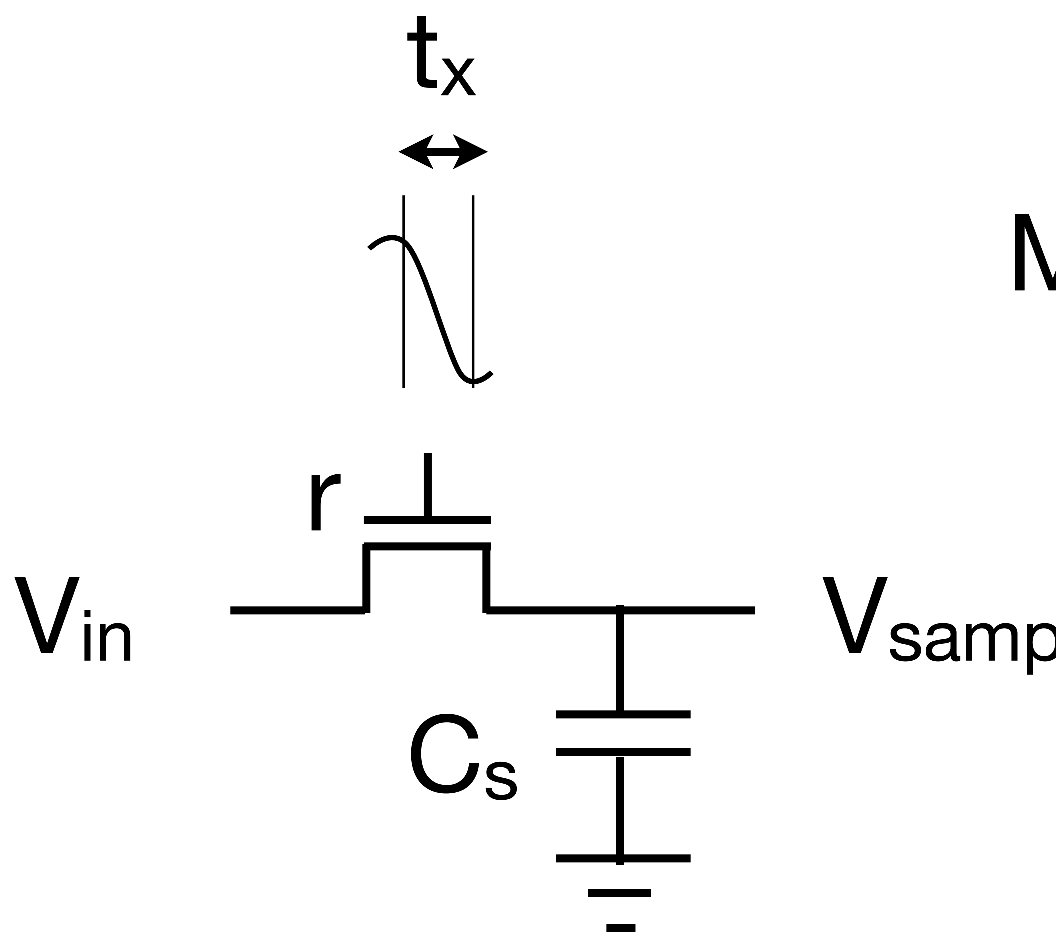
kernel

out



- Lowpass example
- Output is weighted sum of recent inputs

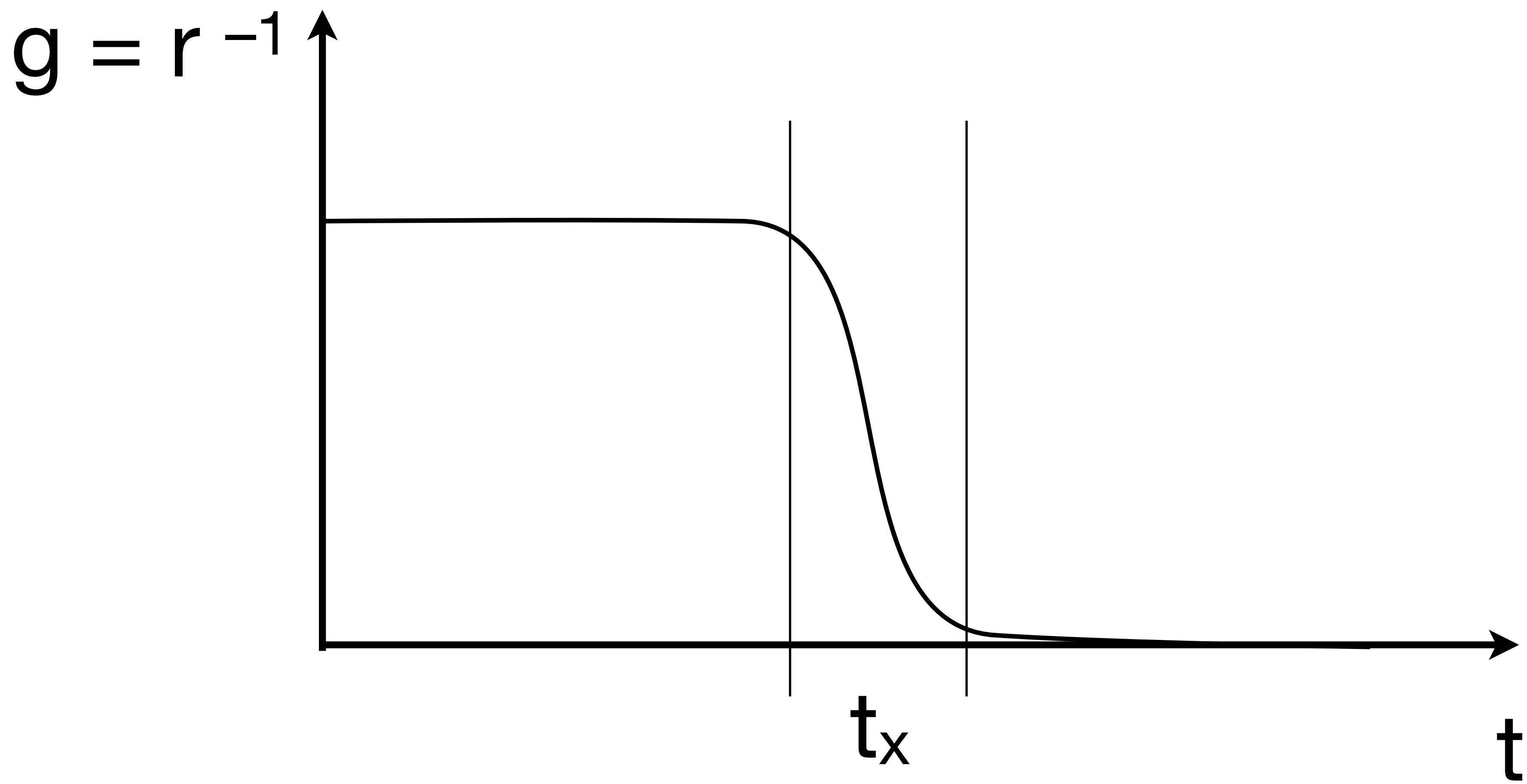
Somewhat more realistic sampler…



MOS switch

- When is the input value actually sampled?

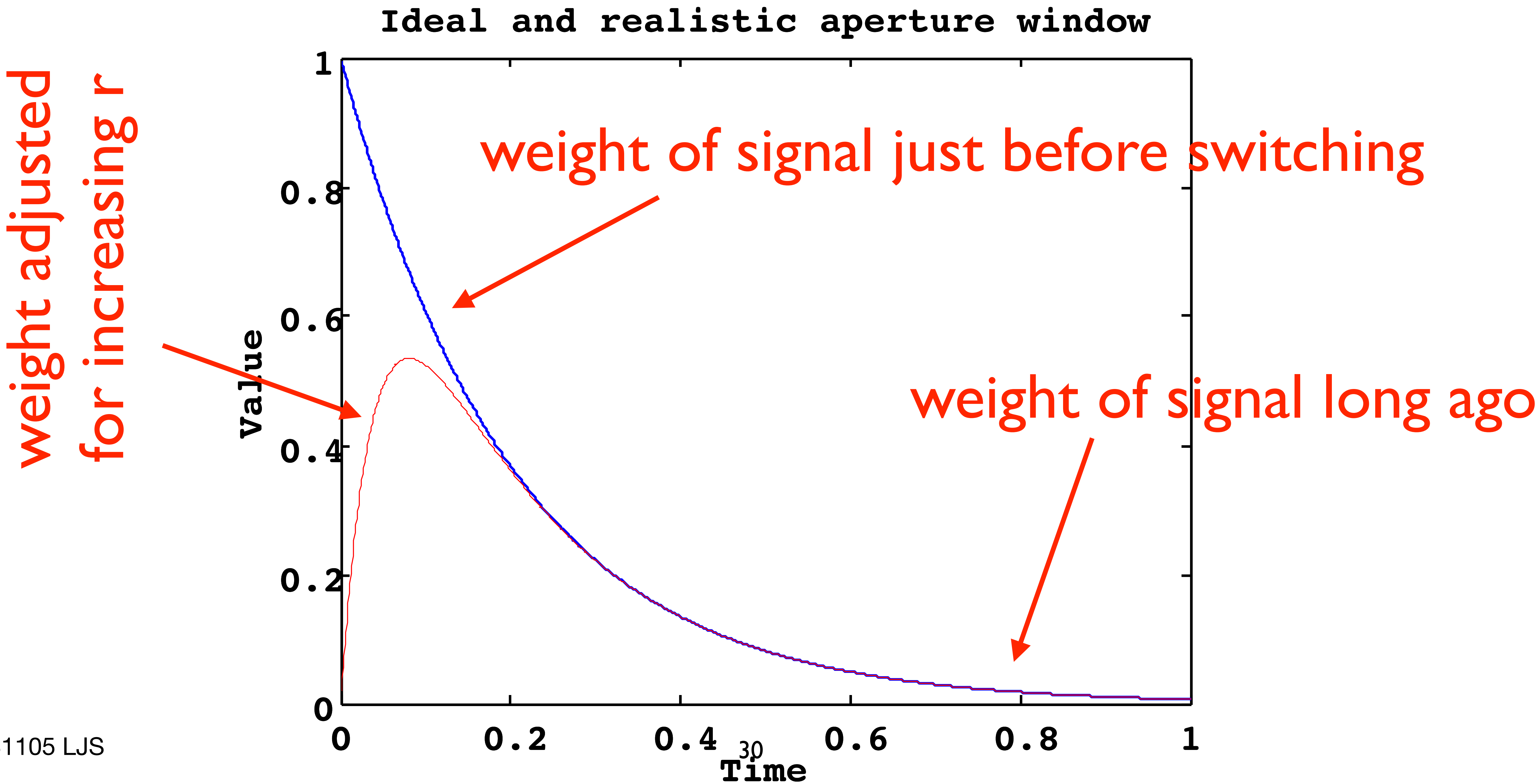
$g_{switch}(t)$



- When switch opens, r approaches infinity, so $g = r^{-1}$ approaches 0
- Gradual switching over interval t_x

Aperture window limits accuracy

- Switch resistance r grows gradually over t_x
- Window differs from RC response!

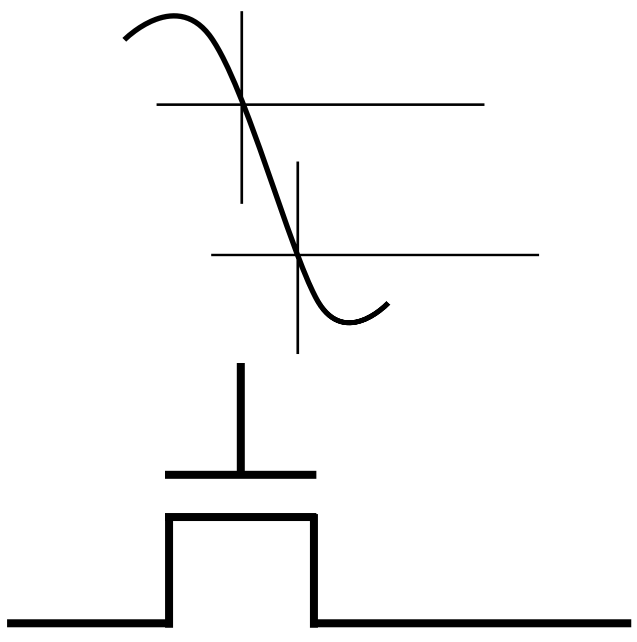


But why does this matter?

- Aperture window duration must be smaller than sample interval, so therefore the integration time is small enough not to affect sampled value much? Right?
- No. Not if undersampling is used.
  - $f_{sig} > f_S / 2$
  - Higher Nyquist band!

Aperture window summary

- Aperture window low-pass-filters the signal
- Especially significant when undersampling



- Also, switch is never really linear
  - Aperture window function depends on voltage!

Distortion!

Summary

- Sampling intended to give one-to-one mapping from continuous to discrete domain
  - Frequency aliasing may defeat this intention
  - Anti-alias filters suppress out-of-band signals (but not perfectly)
- Aperture window acts as low-pass filter
- Switch non-linearities cause distortion

Next lecture:
non-uniform sampling
(jitter)