Discrete-time filters

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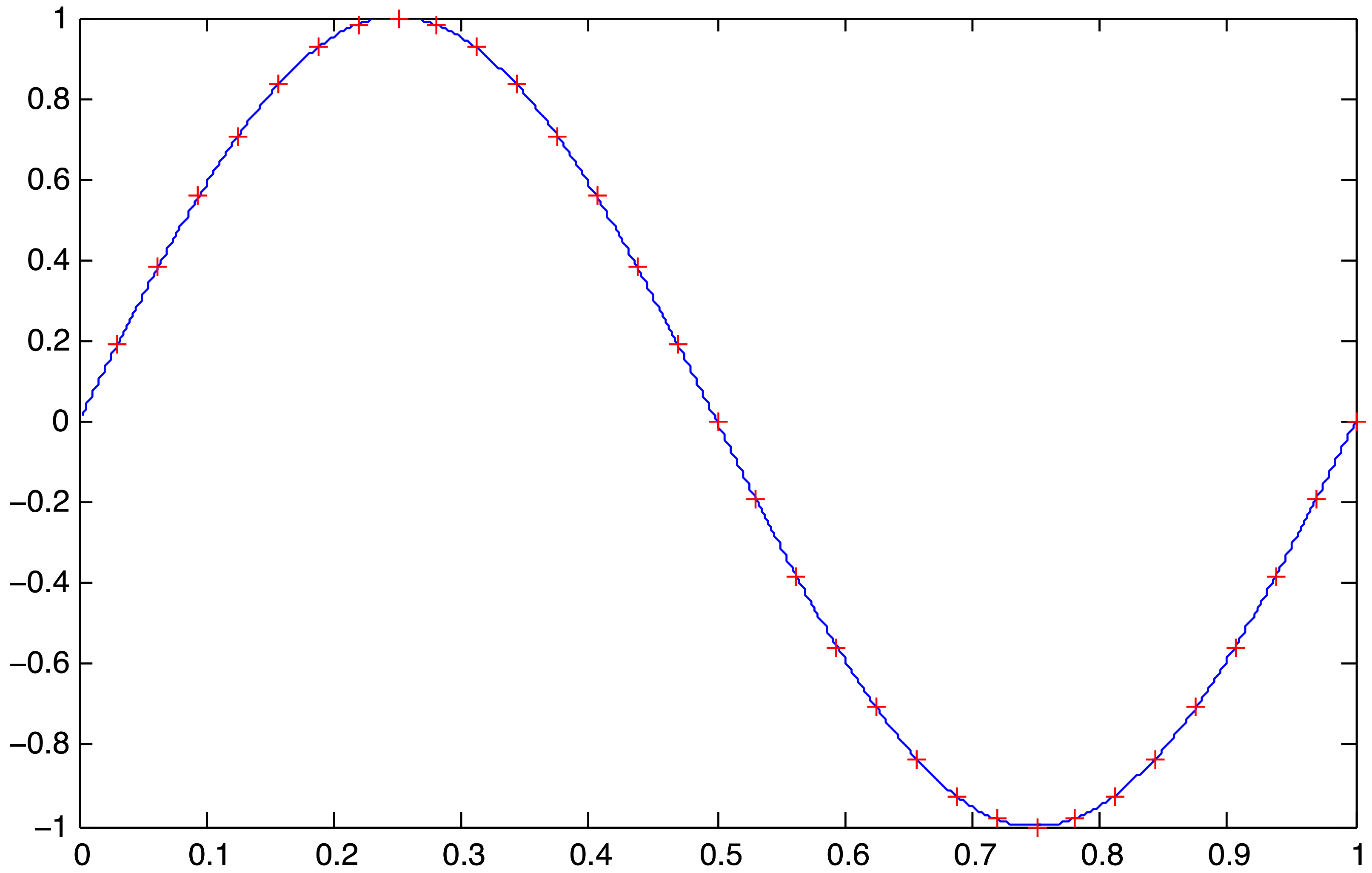
Topics for today

- Discrete-time signals (again)
- Discrete-time filters
    - Z transform (cf. Laplace transform)
- Example: Multirate processing

Discrete-time signals



recap slide

$f_s = 32$

$y = \sin(2\pi \cdot t),\ 0 < t \leq 1$

$y = \sin(2\pi \cdot t_k),\ t_k = (k / f_s)$

Discrete vs continuous

- Sampling gives discrete-time version of continuous-time signal
- Identical values at sequence of time instances
    - Ideally, instances are uniformly spaced
- One-to-one correspondence possible for band-limited continuous-time signal
    - Nyquist criterion

Continuous-time filters

- Linear, time-invariant (LTI) systems
    - Described by system of differential-algebraic equations (DAEs)
    - Define LTI relation between output(s) and input(s)
- Time or frequency domain (equivalent)
    - Time: convolve w/ impulse response
    - Frequency: multiply with transfer function

Restrictions

- Lumped components
  - R, L, C, gains
- Then, <u>limited-order</u> differential equations
  - Example: first-order DE for simple RC link
- Then, frequency domain representation is <u>rational</u> function of same order
  - Poles, zeroes defined by two polynomials

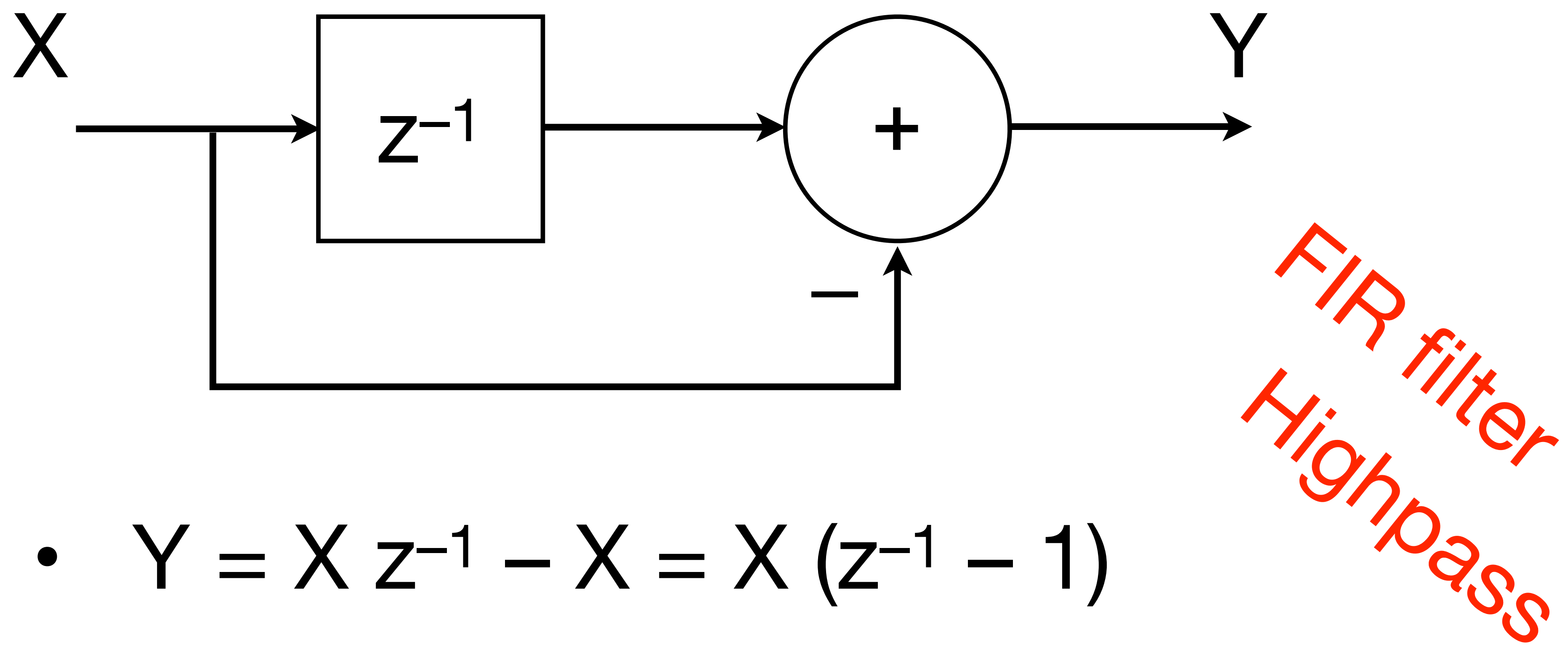
Discrete-time filters

- Similar to continuous-time filters…
  - … if you sample fast enough…
- Z transform replaces the Laplace transform in the sampled domain
  - Simple rules for constructing transforms, as in Laplace case
  - Used also for digital filters (after quantization)

Z vs. Laplace

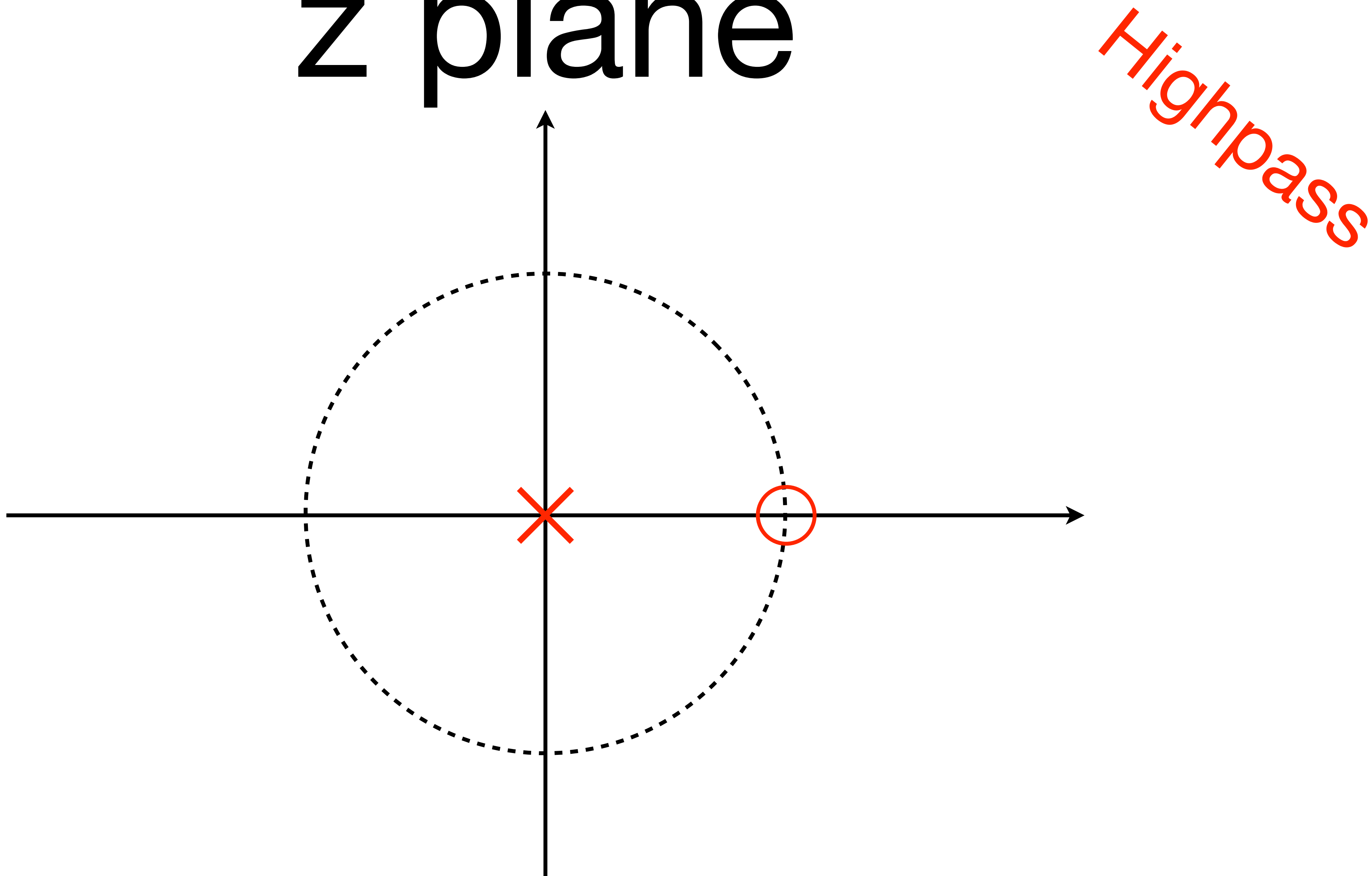
- Laplace:
  - s^{-1} means integration
  - Transfer functions rational in s: Z(s) / P(s)
- Z:
  - z^{-1} means delay by one sample interval
  - Transfer functions rational in z: Z(z) / P(z)

Simple filter in z domain



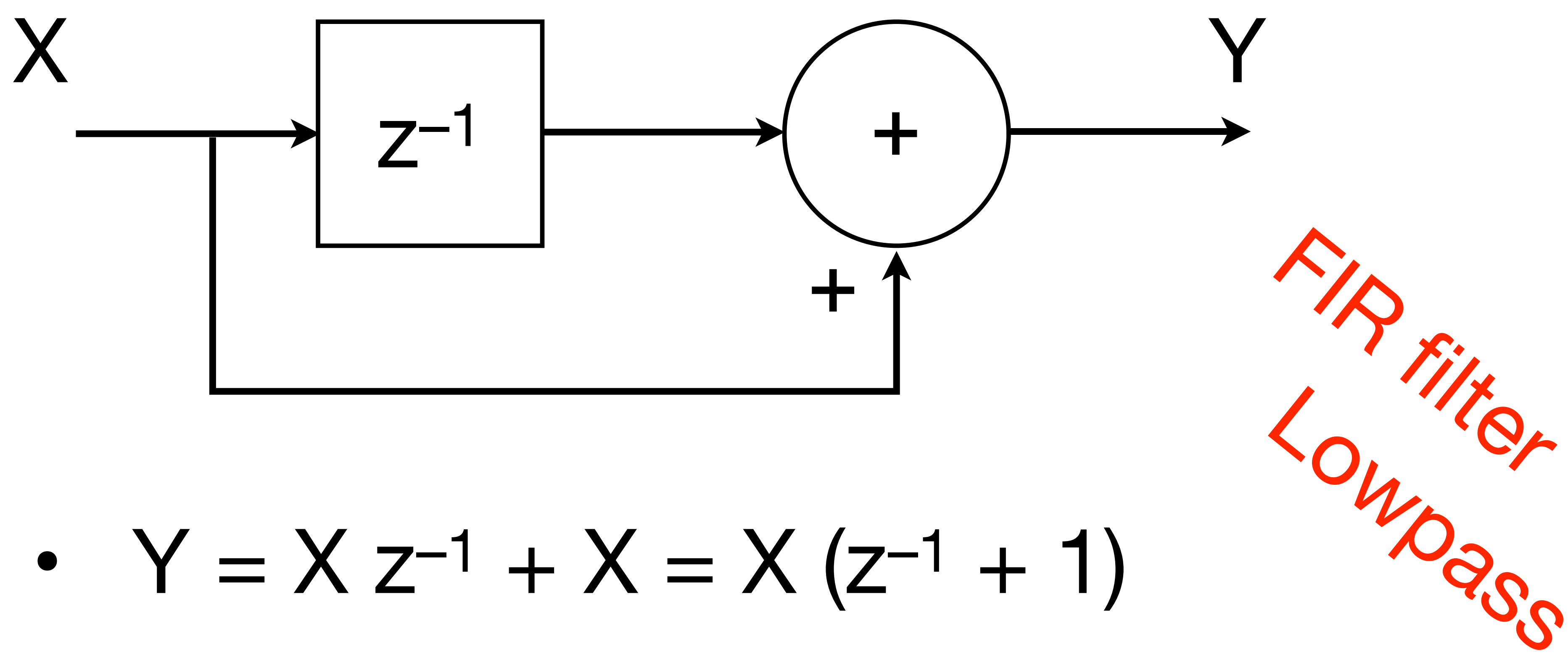
- $Y = X z^{-1} - X = X (z^{-1} - 1)$
- $Y / X = (z^{-1} - 1) = (1 - z) / z$
- Poles? Zeros?
- DC gain?

z plane



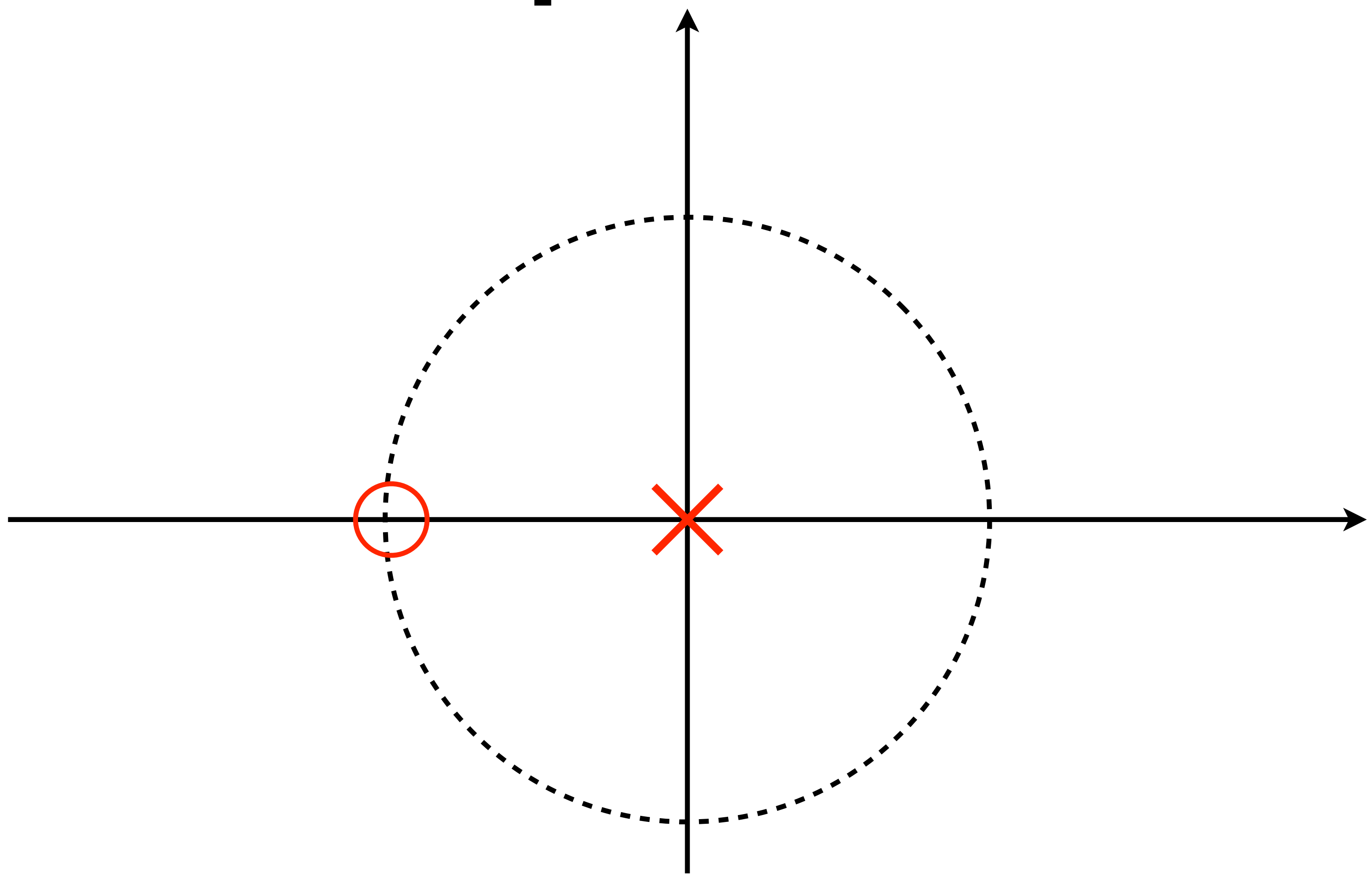
- $Y / X = (1 - z) / z$
- DC gain = 0 means a zero at z = 1
- Pole at z = 0

Simple filter in z domain (2)



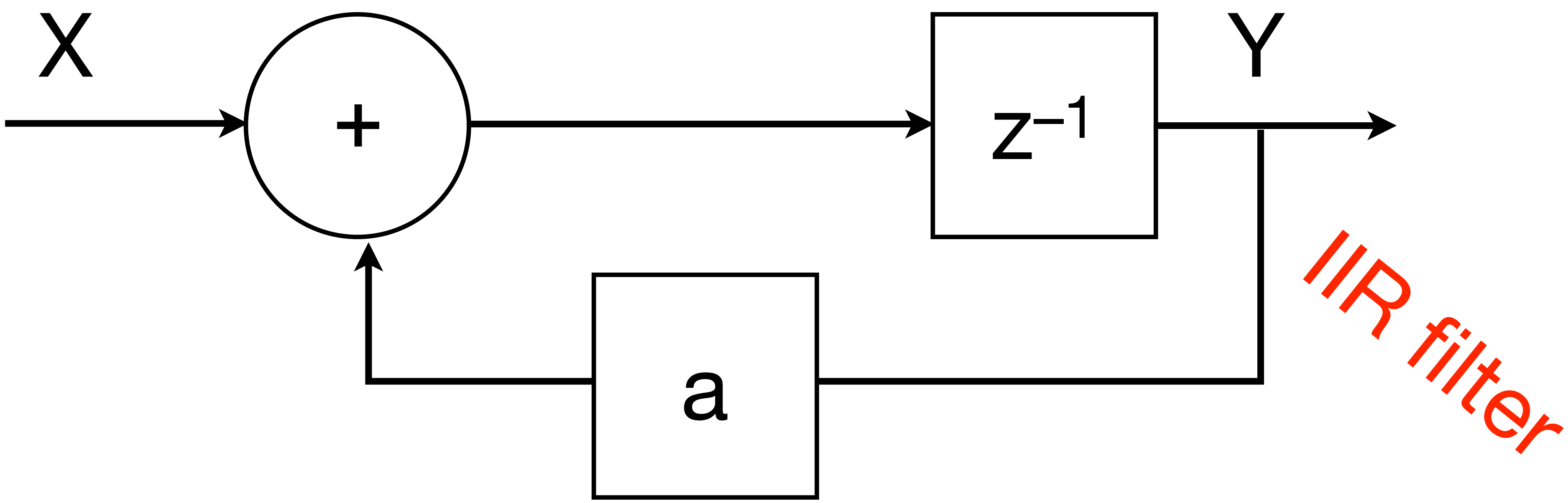
- $Y = X z^{-1} + X = X (z^{-1} + 1)$
- $Y / X = (z^{-1} + 1) = (1 + z) / z$
- Poles? Zeros?
- DC gain? Gain at $f = 1 / (2 T_S)$?

z plane



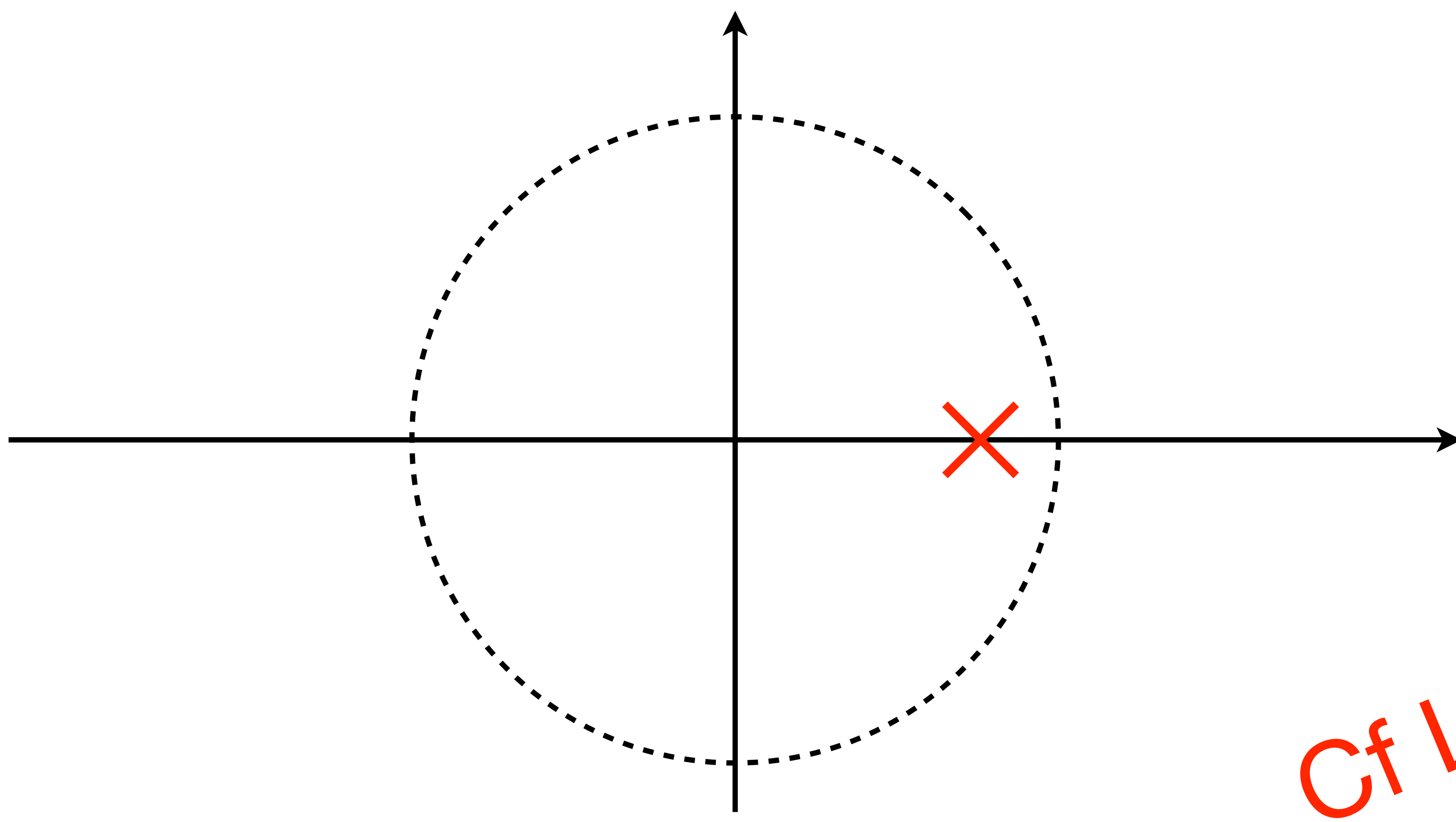
- $Y / X = (1 + z) / z$
- Pole at $z = 0$
- Zero at $z = -1$ means gain = 0 at $F_S / 2$

Simple filter in z domain (3)



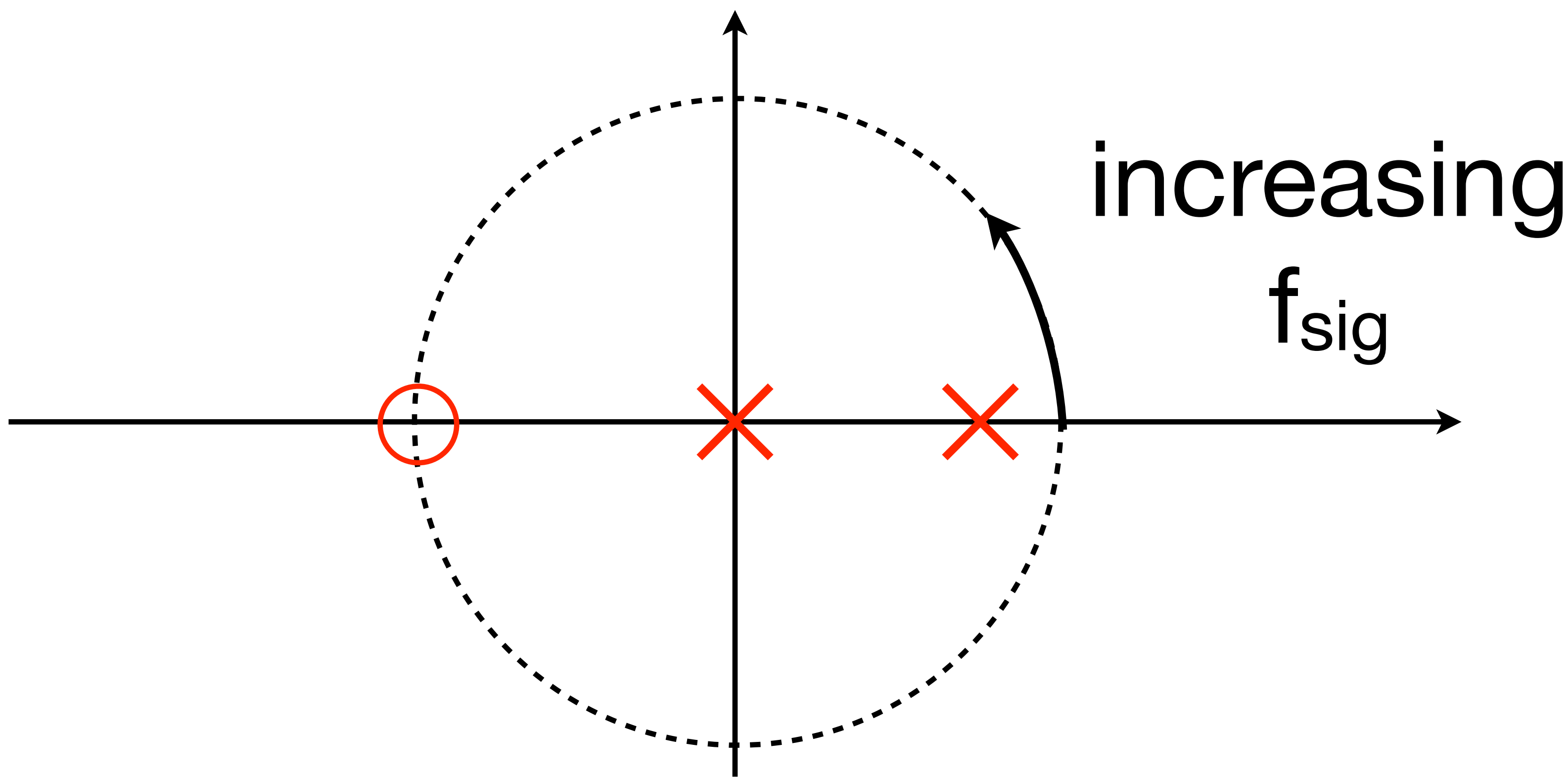
- $Y = z^{-1} (X + aY)$
- $Y / X = z^{-1} / (1 - az^{-1}) = 1 / (z - a)$
- Poles? Zeros?
- Stability?

z plane



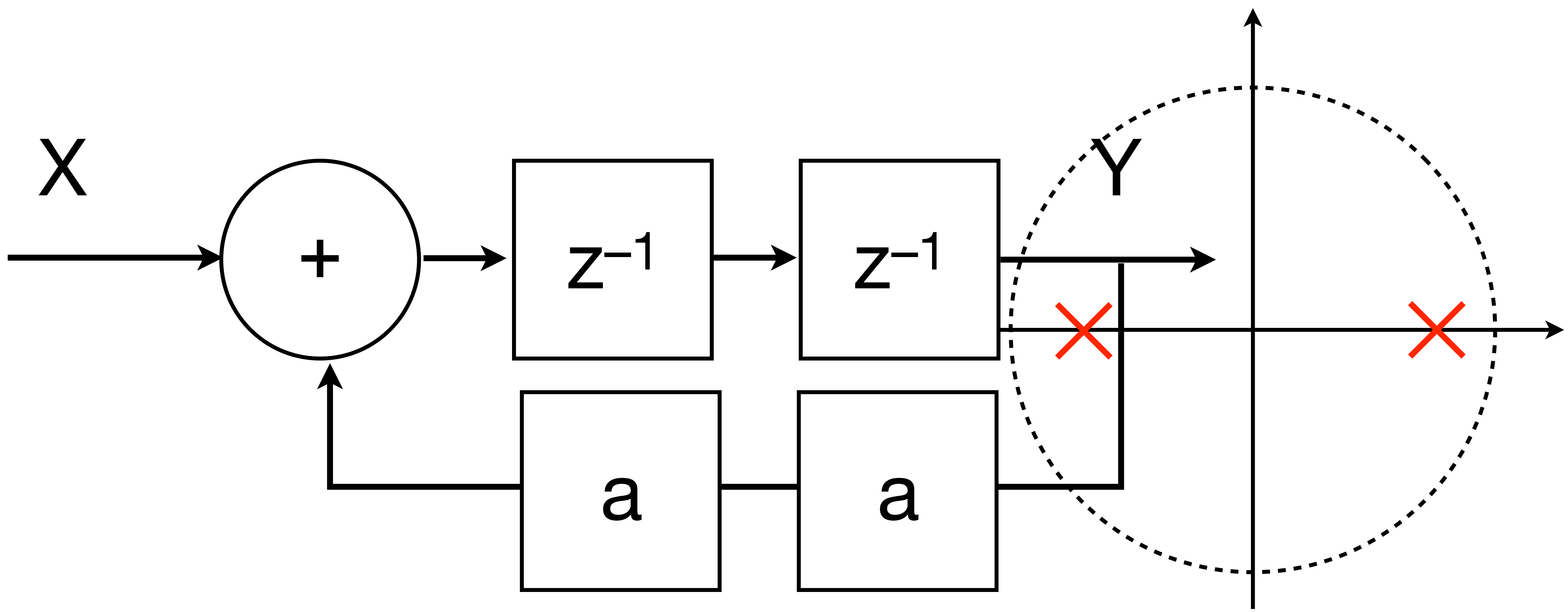
- $Y / X = 1 / (z - a)$
- Pole at $z = a$
- Stable for $|a| < 1$ (in general, stable if poles inside unit circle)

Frequency response



- Move along unit circle, evaluate xfer func
- Corresponds to moving along imag. axis in s plane
- Note: xfer func values repeat! Aliasing!

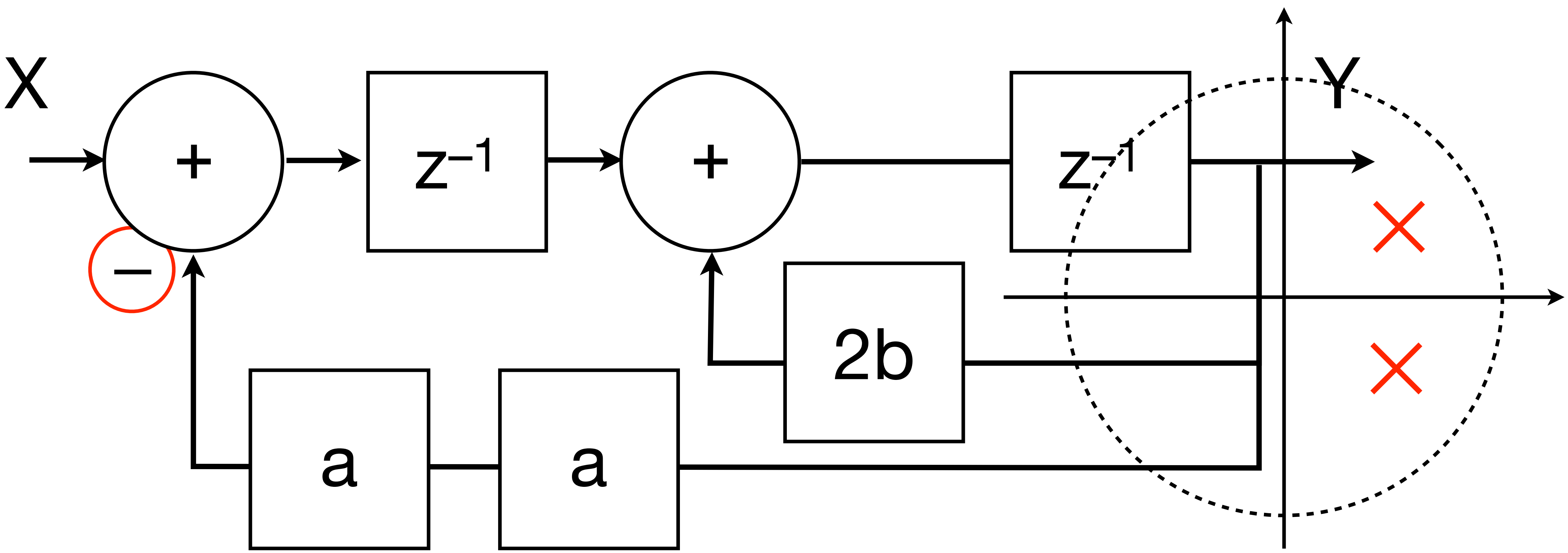
2nd order



- $Y = z^{-1} z^{-1} (X + a^2 Y) = z^{-2} (X + a^2 Y)$
- $Y / X = z^{-2} / (1 - a^2 z^{-2}) = 1 / (z^2 - a^2) =$

$= 1 / ((z + a)(z - a))$

Complex poles



Poles complex if a > b

- $Y = z^{-1}\,2bY + z^{-2}\,(X - a^2Y)$
- $Y / X = z^{-2} / (1 - 2bz^{-1} + a^2z^{-2}) =$

  $= 1 / (z^2 - 2bz + a^2)$

- P-Z placement symmetric around real axis

Design method re-use

- Continuous-time design re-usable in discrete time!
  - Cf. re-use of LP design for HP, etc.
- “Map” s-plane on z-plane
  - Poles appear also in z-plane
- Find mapping which preserves desirable properties

Desirable properties

1. Stable in CT should be stable in DT
   - Map LHP on inside of unit circle
2. CT lowpass should yield DT lowpass, etc
   - Origin should map to 1
3. Other filter properties should be preserved
   - Attenuation in pass- and stopband
   - Flatness in pass- or stopband

Such mappings exist.

Practical DT filter design

- Select transfer function
  - Classical (Butterworth, etc) or purpose-built
- Select implementation style
  - Parameter sensitivity in passband vs stopband etc
- Select “good-enough” circuit implementation
  - Consider inaccuracies

- Very similar to CT filter design!

DT analog vs digital?

- May any digital filter be transferred to DT-analog?
- Well, yes. However:
    - TF zeros require good matching
    - Low-frequency TF poles require low-leakage integrators
    - Noise and distortion differ
    - …

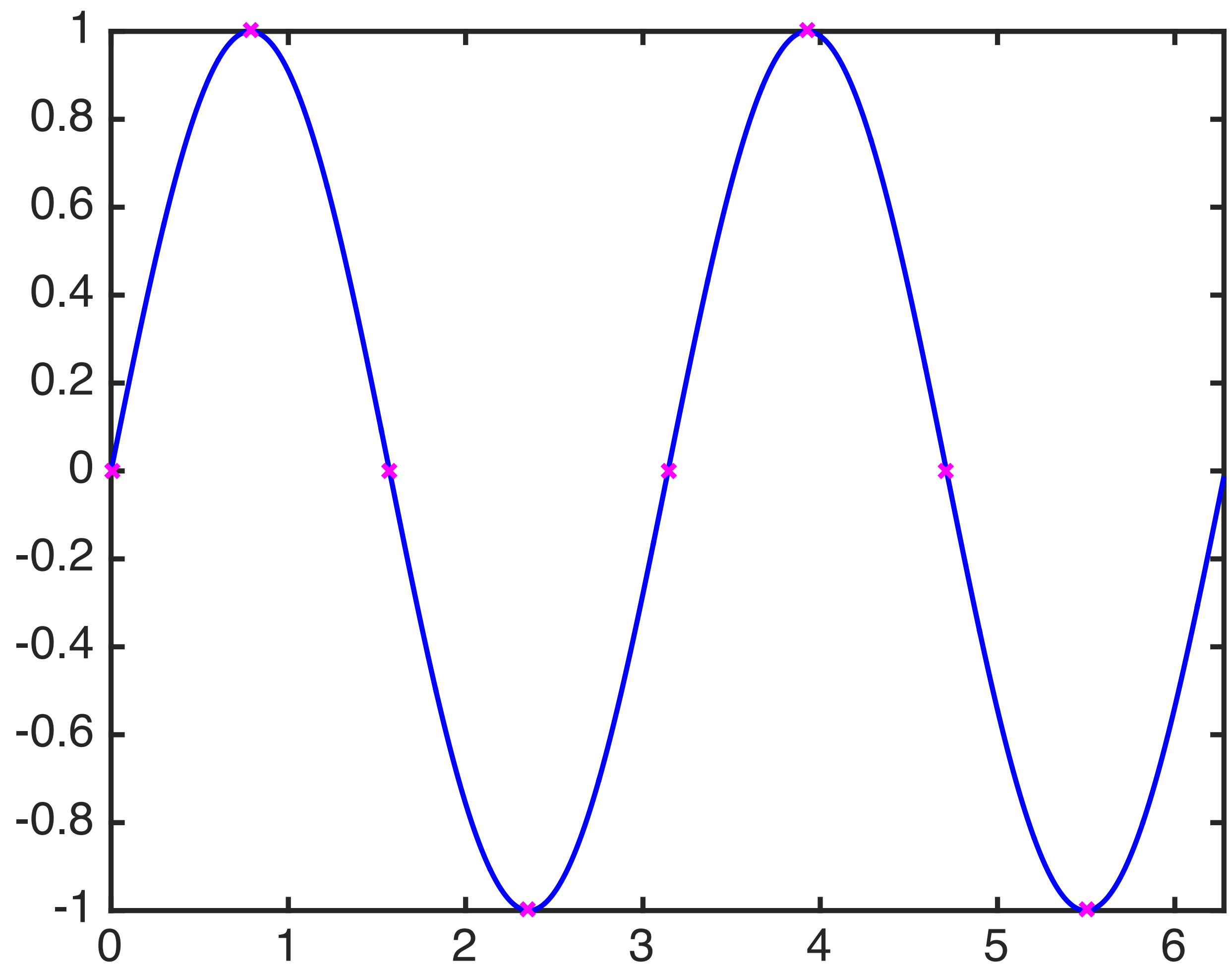
Different tradeoffs
→
different solutions

DT filtering application: Multirate processing

Problem

- Have signal sampled at f_{old}, need signal sampled at f_{new}. How?
- Common practical problem!
  - Example: DAT audio (48 kHz) to CD audio (44.1 kHz)
- Typical requirements:
  - Signal of interest must be retained
  - Noise etc should not increase

1. Integer rate <u>reduction</u>

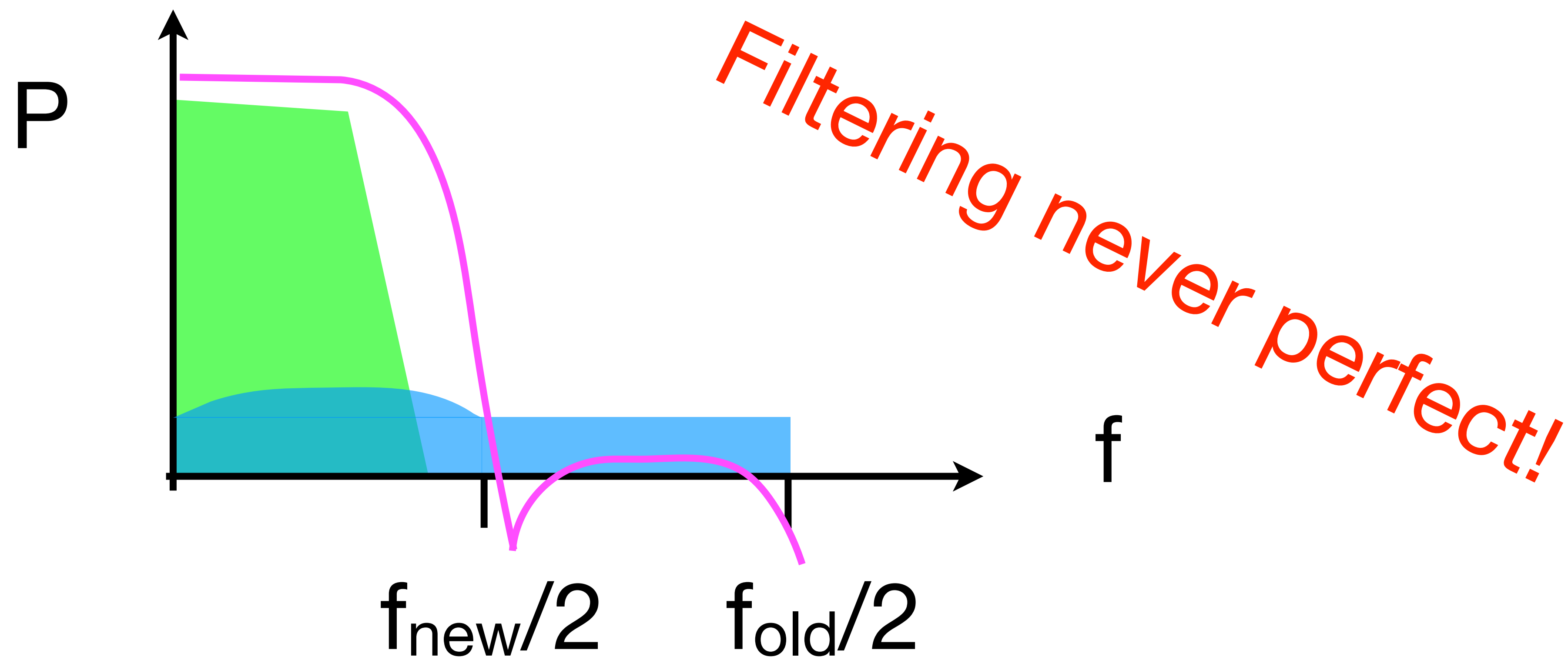


- $f_{new} = f_{old} / N$ (here N = 2): drop some samples!
  - When no loss of information?

Rate reduction conditions

- Nyquist conditions must be valid for new sample rate!
  - No signal components above f_{new} / 2
  - Signal of interest: OK. But what about noise?

Noise folding / aliasing

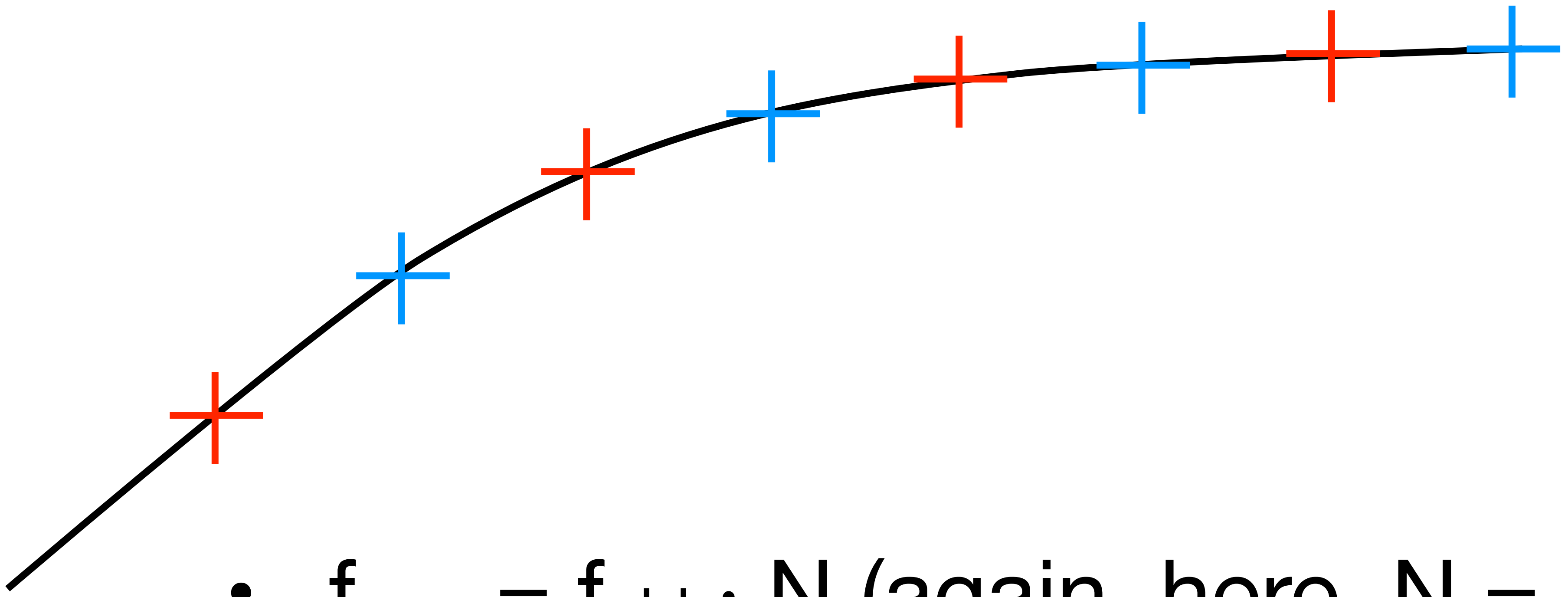


- Signal-of-interest unchanged (Nyquist fulfilled)
- All noise below f_{old} / 2 folded into f_{new} / 2
  - Reduce by <u>filtering before rate change</u>

Optimization

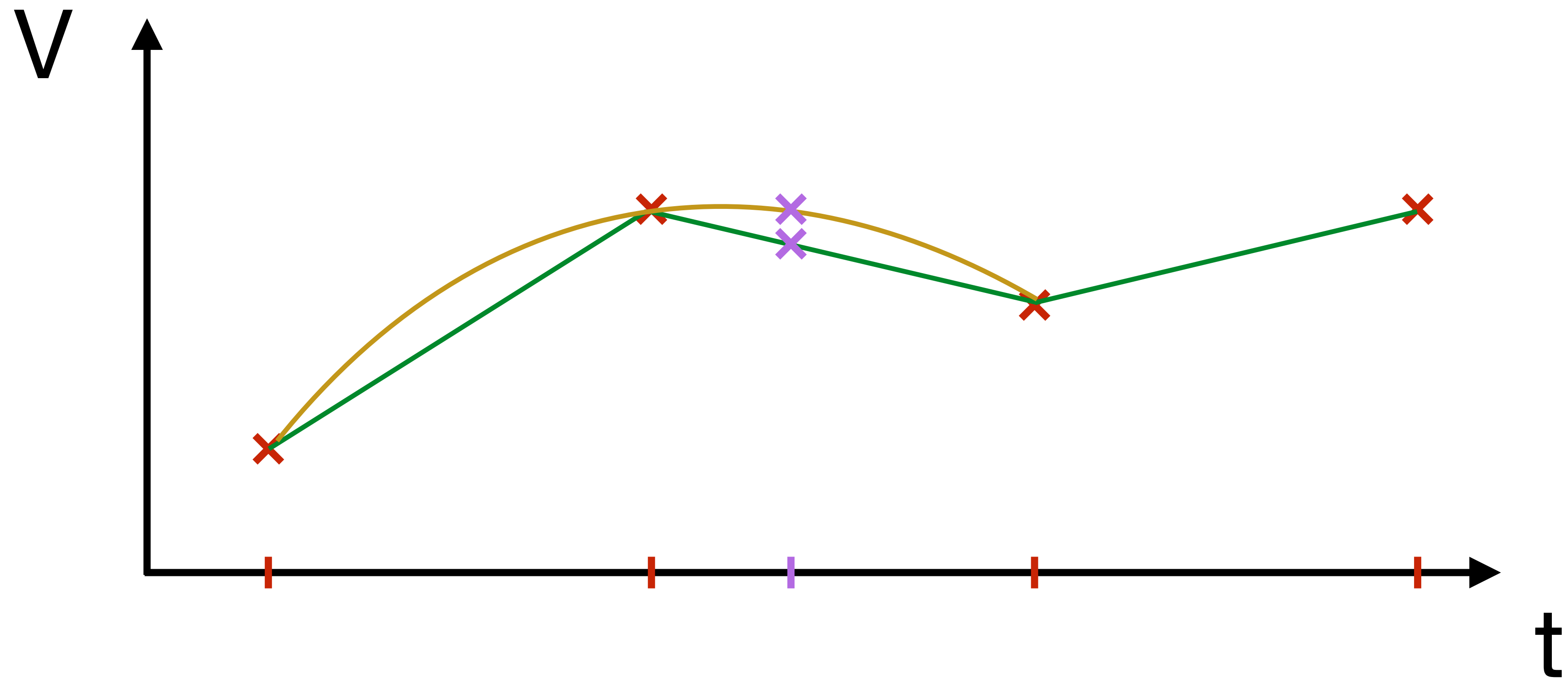
- Single-rate filtering: calculate one new sample for each old sample
  - Sliding average (FIR filter)
  - Recursive definition (IIR filter)
- Rate reduction: N–1 out of N samples will be dropped immediately
  - Don’t ever calculate them!
  - May save factor N of calculations 😀

2. Integer rate increase



- $f_{new} = f_{old} \cdot N$ (again, here, N = 2)
- Interpolate for new samples
- Calculate new samples from old
- How estimate performance?

Interpolation



- Originally used to find values for intermediate values in tables
  - Linear (2 original values → first-order expression)
  - Square (3 values, 2nd-order expression)
  - Polynomial (N values, N–1-order expression)

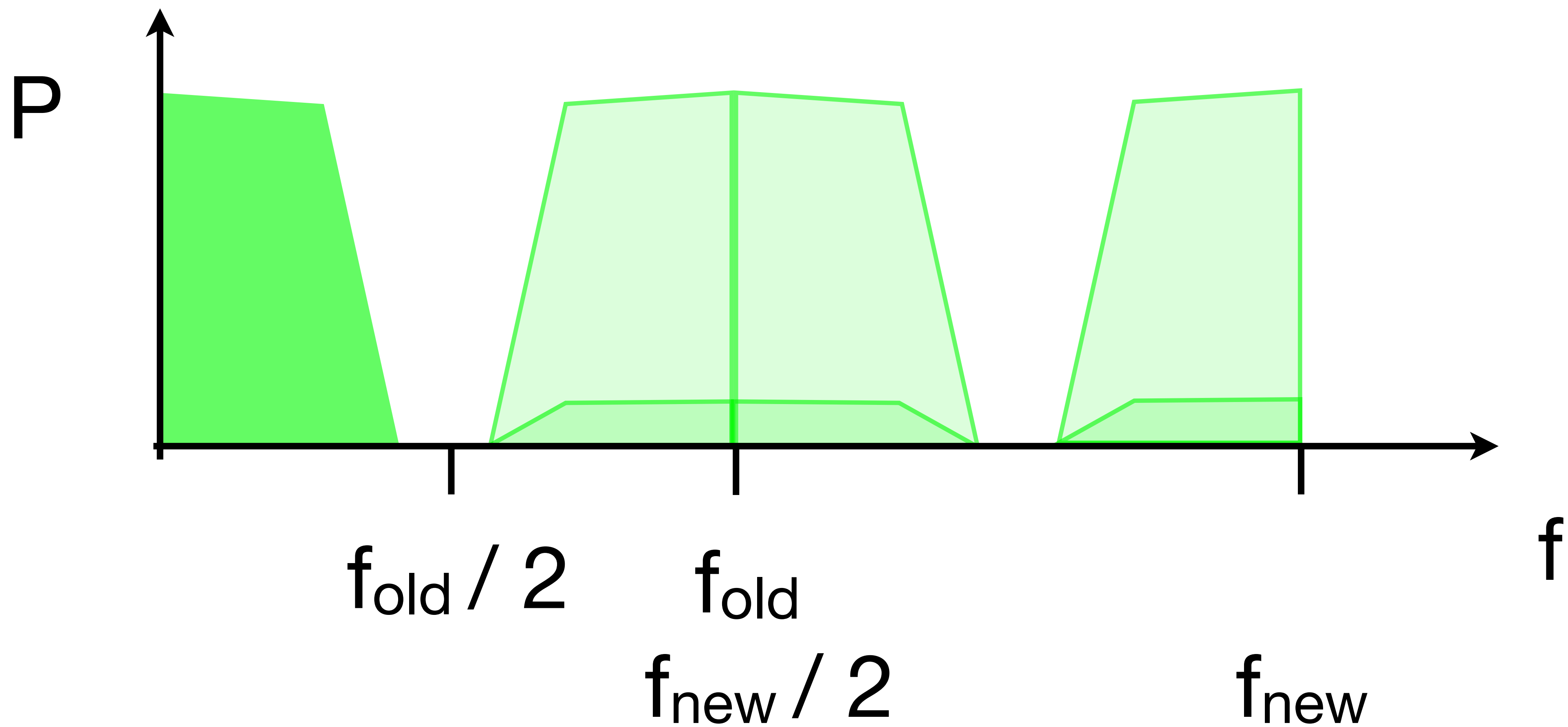
Filtering

Frequency domain vs time domain

- View interpolation as filtering!
  1. Insert intermediate samples with value = 0
  2. Run resulting sequence through low-pass filter
- New sample values derived from old values (inserted ones are 0)
- Filter performance determines accuracy

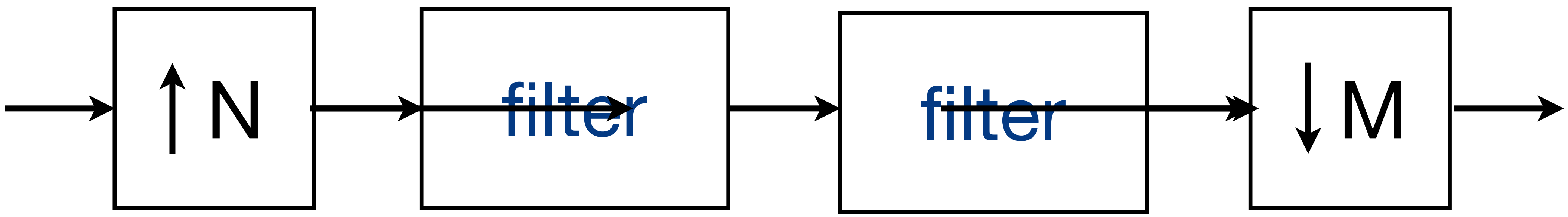
- Again, don't calculate unless necessary

Frequency domain



- Mirror images in continuous-time spectrum
- Must be suppressed by filter
  - Filter specs derived from allowable error

Rational rate change



- $f_{new} = f_{old} \cdot N / M$
- Increase + decrease
- Combine filters

Implementation

- Some ratios require large N, M
- Large N means high intermediate sample rate
  - High dynamic power!
- May be better to use several steps
  - $f_{new} = f_{old} \cdot (N_1 / M_1) \cdot (N_2 / M_2) \cdot \ldots$
  - Co-optimize with filter cost…

DT filter summary

- "Same" theory as for digital filters
- Theory similar to that of CT filters
  - Transfer functions, impulse response, etc
  - Z transform rather than Laplace transform
- CT filter designs can be reused to certain extent
  - Map s plane to z plane
- Sample rate change involves DT filtering