Transients on Transmission Lines

16

Swift as a shadow, short as any dream,
Brief as the lightning in the collied night,

William Shakespeare, A midsummer night's dream

16.1 Introduction

Chapter 14 discussed the propagation properties of transmission lines with particular emphasis on impedance, the reflection coefficient, and time-harmonic representation. Voltage and current were phasors, and a number of properties such as the speed of propagation, wavelength, and phase and attenuation constants were used as a direct consequence of the time-harmonic nature of the waves. Much of the discussion paralleled that of propagation of plane waves in unbounded domains.

There are, however, important applications in which the single-frequency, time-harmonic representation is not appropriate. For example, when we close a switch on a transmission line connecting the line with the generator, a transient ensues. In effect, we are connecting a step source to the line. Similarly, when disconnecting the line, we should expect a transient. When a power transmission line, which may normally operate under steady-state conditions, is shorted because of a fault or when the load suddenly changes, a transient is again generated. In still other cases, such as in digital communication lines, narrow pulses may be sent at relatively high rates. Similarly, the lines connecting digital circuit components on a board transfer pulses which may be wide or narrow, depending on the application. A number of transient waveforms of this type are shown in **Figure** 16.1. In all of these applications, we cannot use the methods of the previous chapters directly. In fact, many of the basic concepts used in the previous chapters are not properly defined in this new environment. For example, the speed of propagation, wavelength, phase constant, and even impedance are only properly defined in the time-harmonic environment.

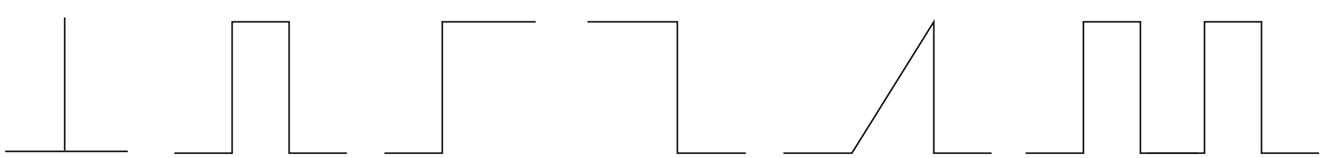

Figure 16.1 Common transients encountered in analog and digital communication lines

The approach adopted here is a very different and fundamental approach. Imagine that we could observe the behavior of the line at all times and at any point we wished. This would give us all the information needed to evaluate the behavior of the line. In effect, we are going to "ride" the various waves that may exist on the line as they propagate. This approach has the great advantage that it is simple and intuitive. It will provide simple solutions to a number of important transmission line applications with few assumptions.

Two types of transients will be discussed here. The first is narrow pulses and the second is the step source. The intermediate case of long pulses will be treated as the superposition of step sources.

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16.2 Propagation of Narrow Pulses on Finite, Lossless Transmission Lines

Narrow pulses are common in digital systems but also on communication lines and are characterized by widths which are very small compared with the propagation time along the line. In other words, if a line is of length d [m] and the speed of propagation is v_p [m/s], the time of propagation on the line is $t_p = d/v_p$ [s]. A pulse of width $\Delta t \ll t_p$ is considered a narrow pulse. Note, however, that Δt itself is not necessarily small.

A narrow pulse propagates on a lossless line without distortion since the speed of propagation is independent of frequency. All frequencies are propagated at the same speed. Thus, we can still use the concept of phase velocity even though it was initially defined for time-harmonic waves. The speed of propagation on the line is

$$v_p = \frac{1}{\sqrt{LC}} \quad \left[\frac{\text{m}}{\text{s}}\right] \tag{16.1}$$

where L and C are the inductance and capacitance per unit length of the line, respectively.

Consider first the line in **Figure 16.2**. The load is matched to the line so there will be no reflection from the load. The generator produces a pulse at time $t = 0$. The pulse appears at the input to the line with the following amplitude for voltage and current:

$$V^+ = V_g \frac{Z_0}{Z_0 + Z_g} \quad [\text{V}], \quad I^+ = \frac{V^+}{Z_0} = \frac{V_g}{Z_0 + Z_g} \quad [\text{A}] \tag{16.2}$$

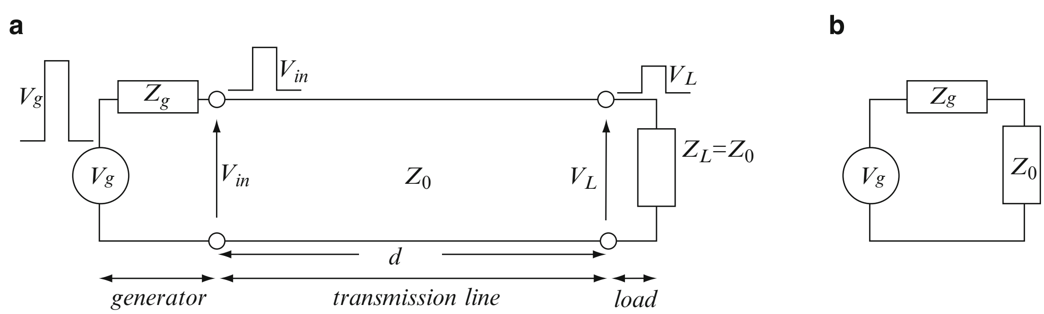


Figure 16.2 (**a**) Propagation of a narrow pulse on a matched line. (**b**) Equivalent circuit at the generator at $t = 0$

This is due to the impedance divider created by the generator's internal impedance and the line impedance. The line current is equal to the forward-propagating voltage divided by the line impedance, which, in this case, equals Z_0 since the pulse has not propagated down the line and the only impedance it sees is the characteristic impedance of the line. This pulse now propagates toward the load, which it reaches after a time $t = d/v_p$. Since the load impedance is equal to the characteristic line impedance, there is no reflection at the load ($\Gamma_L = 0$), and all energy in the forward-propagating pulse is transferred to the load. Nothing more happens on the line unless additional pulses are generated.

Now suppose the line is not matched, as shown in **Figure 16.3**. At time $t = 0$, a pulse appears at the generator terminals. Since nothing happened on the line itself, the generator only sees the characteristic line impedance. Thus, the initial pulse that appears at the generator's terminals is the same as for the matched line in **Eq. (16.2)**. The pulse propagates at the same speed and reaches the load. The pulse is partly transmitted into the load, but because the line and load are not matched, there is a reflection coefficient at the load:

$$\Gamma_L = \frac{V^-}{V^+} = -\frac{I^-}{I^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{16.3}$$

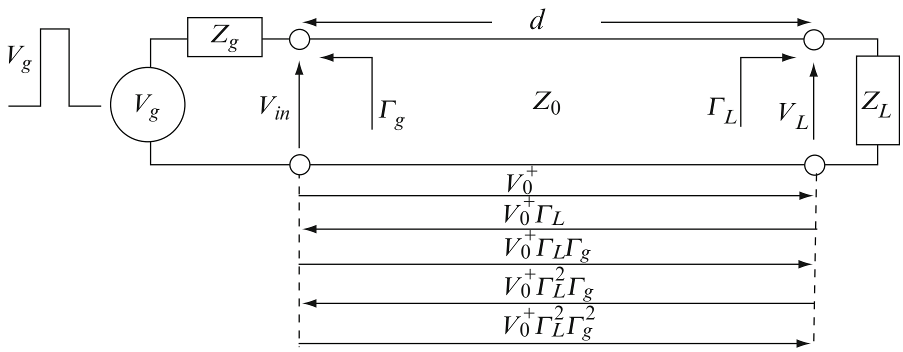


Figure 16.3 Mismatched load and generator. The first few reflected voltages at load and generator are shown

Also, because the sum of the forward and reflected waves must equal the transmitted wave, the transmission coefficient at the load is

$$T_L = 1 + \Gamma_L = 1 + \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{2Z_L}{Z_L + Z_0} \tag{16.4}$$

The reflected voltage and current waves are

$$V_1^- = \Gamma_L V^+ = \frac{Z_L - Z_0}{Z_L + Z_0} V^+ = \frac{Z_0}{Z_0 + Z_g}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) V_g \quad [\text{V}] \tag{16.5}$$

$$I_1^- = -\Gamma_L I^+ = -\frac{V^+}{Z_0}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) \quad [\text{A}] \tag{16.6}$$

The total voltage at the load at time $t = d/vp$ is the sum of the incoming and reflected waves:

$$V_{L1} = V^+ + V_1^- = V^+(1 + \Gamma_L) \quad [\text{V}] \tag{16.7}$$

where the index 1 indicates that this is the first reflection at the load. Note that although a sum is used, the reflection coefficient can be negative. The current in the load is given from **Eq. (16.3)** as

$$I_{L1} = \frac{V^+}{Z_0}(1 - \Gamma_L) \quad [\text{A}] \tag{16.8}$$

The sum of the forward- and backward-propagating waves only exists for a period equal to the width of the pulse. After that, only the backward-propagating waves in **Eqs. (16.5)** and **(16.6)** exist on the line. To see how this comes about, the forward-propagating wave and the backward-propagating wave can be viewed as two separate waves propagating in opposite directions, as shown in **Figure 16.4a**. For clarity, we assume that Γ_L is negative, but it may also be positive. After $t = t_1$, the pulses add up as shown by the solid lines in **Figure 16.4b**. At a time $t > t_1 + \Delta t$, the only wave on the line is the backward-propagating wave, as shown in **Figure 16.4c**.

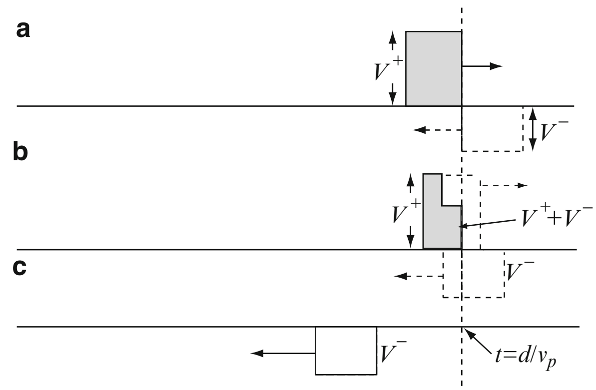


Figure 16.4 Conditions at the load before, during, and after reflection. (**a**) The pulse front reaches the load. (**b**) A reflected wave is generated and propagates toward the generator, partially overlapping the incident pulse. (**c**) After one pulse width, only the backward-propagating pulse is left

The reflected voltage (or current) now travels back and, after an additional time equal to d/v_p, reaches the generator. However, now the generator does not act as a generator but rather like a load Z_g since the source of the reflected wave is at the actual load. As with the load, part of the wave is reflected and part is transmitted into the generator (where it must be dissipated). Thus, the backward-propagating wave is reflected into a new, forward-propagating wave at the generator, with the generator reflection coefficient:

$$\Gamma_g = \frac{V_1^+}{V_1^-} = -\frac{I_1^+}{I_1^-} = \frac{Z_g - Z_0}{Z_g + Z_0} \tag{16.9}$$

The reflected waves at the generator are

$$V_1^+ = \Gamma_g V_1^- = \Gamma_L \Gamma_g V^+ \quad [\mathrm{V}] \quad \text{and} \quad I_1^+ = -\Gamma_g I_1^- = \frac{\Gamma_L \Gamma_g V^+}{Z_0} \quad [\mathrm{A}] \tag{16.10}$$

and the total voltage and current at the generator connections are

$$V_{in1} = V_1^- + V_1^+ = V^+ \Gamma_L (1 + \Gamma_g) \quad [\mathrm{V}] \quad \text{and} \quad I_{in1} = I_1^- + I_1^+ = -I^+ \Gamma_L (1 - \Gamma_g) \quad [\mathrm{A}] \tag{16.11}$$

Again, these sums only exist during a time Δt. After that, only the new forward-propagating wave exists. This process repeats itself indefinitely, with each reflection at each end of the line being viewed as a new wave propagating toward the other end. The reflection process is shown schematically in **Figure 16.3** for a few voltage reflections.

If instead of a single pulse, the generator produces a train of pulses, each pulse is reflected as described above. However, both forward-propagating and backward-propagating pulses may meet along the line. When this happens the voltage and current on the line are superposition of the various pulses. Each pulse continues to travel as if it were alone on the line.

Example 16.1 The generator in **Figure 16.5** produces 10 V pulses that are 20 ns wide. Consider a single pulse, produced at $t = 0$. Calculate the voltage and current at the load for all times between zero and 5.5 μs. Assume the line is lossless and speed of propagation on the line is $c/3$ [m/s].

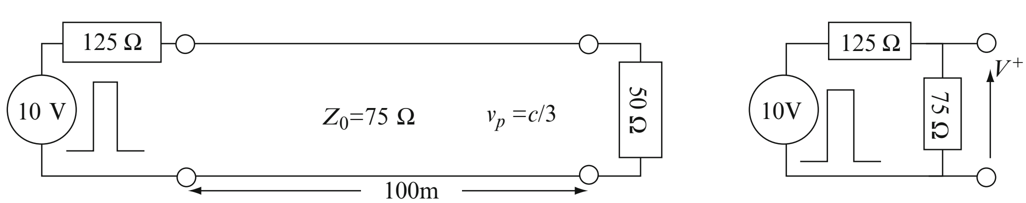


Figure 16.5 A line with mismatched load and generator

Solution: The reflection coefficients at the load (looking into the load) and generator (looking into the generator, from the load) are first calculated. Then, we follow the pulse, based on the time of propagation between generator and load. The time it takes the pulse to travel from the generator to load is

$$t = \frac{L}{v_p} = \frac{100}{1 \times 10^8} = 1 \quad [\mu s]$$

The reflection coefficients at the load and generator are

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - 75}{50 + 75} = -0.2, \quad \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{125 - 75}{125 + 75} = 0.25$$

The voltage and current at the generator at $t = 0$ are

$$V^+ = V_g \frac{Z_0}{Z_0 + Z_g} = V_g \frac{75}{75 + 125} = 0.375 V_g = 3.75 \quad [V]$$

$$I^+ = \frac{V_g}{Z_0 + Z_g} = \frac{10}{75 + 125} = 0.05 \quad [A]$$

These propagate toward the load. After 1 μs, both reach the load. The reflected waves are $V_1^- = \Gamma_L V^+$ and $I_1^- = -\Gamma_L I^+$:

$$V_1^- = V^+ \Gamma_L = -0.2 V^+ = -0.75 \ [V], \quad I_1^- = -I^+ \Gamma_L = 0.2 I^+ = 0.01 \quad [A]$$

The forward- and backward-propagating waves add up for 20 ns at the load. For these 20 ns, the voltage at the load is 0.8 $V^+ = 3$ V and the current is 1.2 $I^+ = 0.06$ A. Both reflected waves propagate back to the generator where a second reflection takes place but now with the reflection coefficient of the generator:

$$V_1^+ = V_1^- \Gamma_g = V^+ \Gamma_L \Gamma_g = 0.25 \times (-0.2) V^+ = -0.1875 \quad [V]$$

$$I_1^+ = I_1^- \Gamma_g = I^+ \Gamma_L \Gamma_g = -0.2 \times 0.25 \times I^+ = -0.0025 \quad [A]$$

Again, at the generator, the voltage is the sum of the backward- and forward-propagating waves for 20 ns. The process now repeats itself with the new forward-propagating waves. At $t = 3$ μs, we are at the load:

$$V_2^- = V_1^+ \Gamma_L = -0.2 V_1^+ = 0.0375 \quad [V], \quad I_2^- = -I_2^+ \Gamma_L = 0.2 I_1^+ = -0.0005 \quad [A]$$

At $t = 4$ μs, the voltage at the generator is

$$V_2^+ = V_2^- \Gamma_g = 0.009373 \quad [V], \quad I_2^+ = -I_2^- \Gamma_g = 0.000125 \quad [A]$$

At $t = 5$ μs, the voltage at the load is

$$V_3^- = V_2^+ \Gamma_L = -0.001875 \quad [V], \quad I_3^- = -I_2^+ \Gamma_L = 0.000025 \quad [A]$$

The results are shown in **Figures 16.6a** and **16.6b** for the voltage and current at the generator and load. The sums of the forward and backward waves are shown.

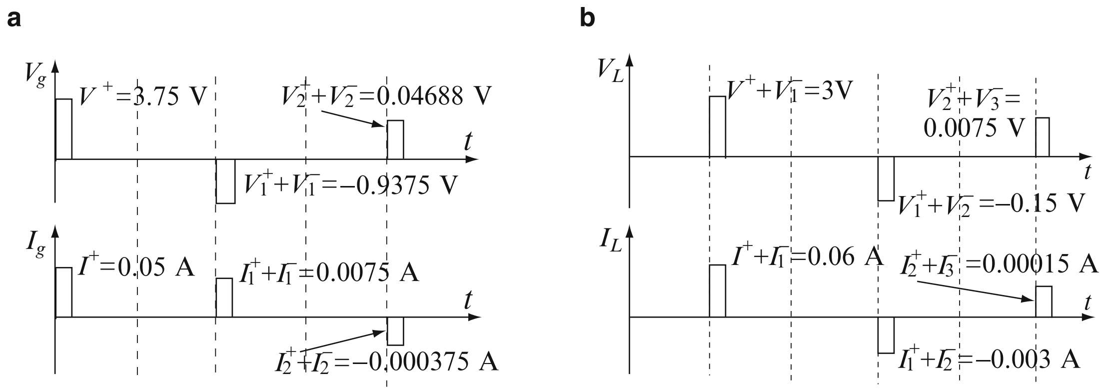


Figure 16.6 (**a**) Voltage and current at the generator in **Figure 16.5**, immediately after the pulses are generated. (**b**) Voltage and current pulses at the load in **Figure 16.5**

16.3 Propagation of Narrow Pulses on Finite, Distortionless Transmission Lines

Although we now assume the line to be lossy, with an attenuation constant α, the line is also assumed to be distortionless (i.e., $R/L = G/C$) so that pulses do not distort. For a single pulse as described in the previous section, all aspects of propagation remain the same, but, in addition, the pulse magnitude is attenuated exponentially as it propagates from generator to load, or load to generator. The problem analyzed here is shown in **Figure 16.7a**.

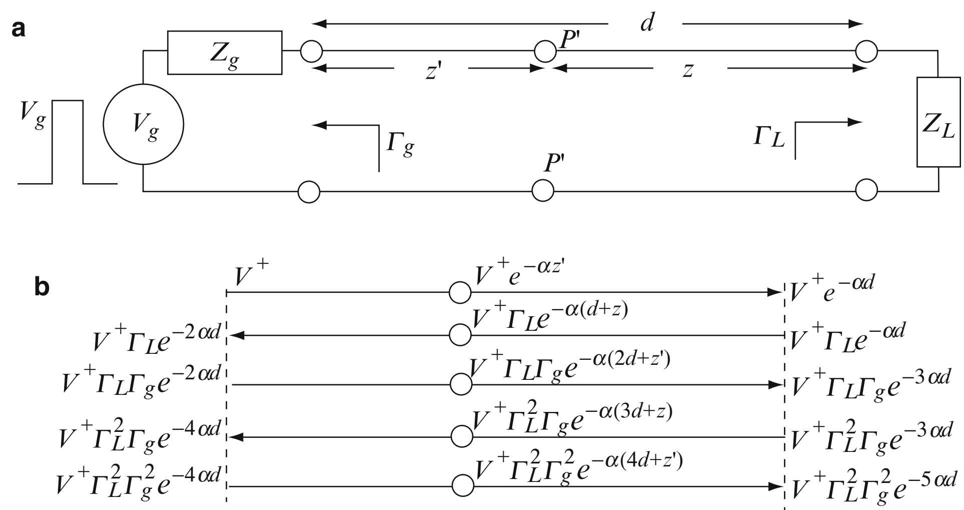


Figure 16.7 (**a**) Distortionless transmission line. (**b**) The voltage waves on the line for a few reflections

With the forward-propagating wave in **Eq. (16.2)**, the wave propagates along the line and is attenuated. For the first wave ($0 < t < d/vp$), the voltage on the line at a point P' is

$$V(z) = V^+e^{-\alpha z'} = V_g \frac{Z_0}{Z_0 + Z_g} e^{-\alpha z'} \quad [\text{V}] \tag{16.12}$$

where z' is the distance from generator to point P' in **Figure 16.7a**. At the load, the forward-propagating wave is

$$V_L^+ = V^+e^{-\alpha d} \quad [\text{V}] \tag{16.13}$$

The reflected wave is

$$V_1^- = \Gamma_L V^+ e^{-\alpha d} \quad [\text{V}] \tag{16.14}$$

At the load, the total voltage is the sum of this and the reflected voltage. This gives

$$V_L = V^+ e^{-\alpha d}(1 + \Gamma_L) \quad [\text{V}] \tag{16.15}$$

However, this sum only exists for a time equal to the pulse width Δt. The reflected wave in **Eq. (16.14)** propagates back and is attenuated. The expression for the reflected wave anywhere on the line between load and generator is

$$V_1^-(z) = \Gamma_L V^+ e^{-ad} e^{-az} \quad [\text{V}] \tag{16.16}$$

This reflected wave reaches the generator and is reflected at the generator unless the generator is matched. At the generator, the first reflection is

$$V_1^-(z = d) = V^+ e^{-2ad} \Gamma_L \quad [\text{V}] \tag{16.17}$$

Taking into account the generator reflection coefficient Γ_g, the total voltage at the generator connections is

$$V_{g1} = V^+ \Gamma_L e^{-2ad}\left(1 + \Gamma_g\right) \quad [\text{V}] \tag{16.18}$$

This sum also exists for a period Δt. The new forward-propagating wave after the first reflection at the generator is

$$V_1^+\left(z'\right) = V^+ e^{-2ad} e^{-az'} \Gamma_L \Gamma_g \quad [\text{V}] \tag{16.19}$$

Thus, the attenuation depends on the total distance traveled by the wave, regardless of how many reflections it has undergone. This is shown schematically in **Figure 16.7b**. Note, also, that each pulse is assumed to travel independently of any other pulses on the line. If two pulses meet anywhere on the line, then the voltage and current at that point and time is the superposition of the pulses. This applies particularly to the location of the load and generator, since for any pulse width, the reflected and incident pulses overlap during a time equal to the pulse width. A sum of more than one pulse may exist on the line at other locations if multiple pulses exist on the line and propagate independently.

Example 16.2 Consider, again, **Example 16.1**, but now the line has an attenuation constant $\alpha = 0.002$ Np/m. Draw the voltage and current at the generator for $0 < t < 5.5$ μs.

Solution: From the above discussion, the voltages and currents at any given time are those for the lossless line multiplied by the attenuation from $t = 0$ to the time considered. Thus, from the results in **Example 16.1**, the voltage and current at the generator only exist at times $t = 0$, $t = 2$ μs, and $t = 4$ μs. At $t = 0$, the waves have not propagated. Thus

$$V^+ = 3.75 \quad [\text{V}], \qquad I^+ = 0.05 \quad [\text{A}]$$

At time $t = 2$ μs, the waves at the generator are V_1^-, I_1^-, V_1^+ and I_1^+. These are attenuated as if they propagated a distance of 200 m. Thus,

$$\begin{aligned} V_1^- &= -0.75e^{-0.002\times200} = -0.50274 \quad [\text{V}], \\ I_1^- &= 0.01e^{-0.002\times200} = 0.0067 \quad [\text{A}] \\ V_1^+ &= -0.1875e^{-0.002\times200} = -0.1257 \quad [\text{V}], \\ I_1^+ &= -0.0025e^{-0.002\times200} = -0.001676 \quad [\text{A}] \end{aligned}$$

At $t = 4$ μs, at the generator, the total distance traveled by the wave is 400 m. The waves at this time are V_2^-, I_2^-, V_2^+ and I_2^+:

$$\begin{aligned} V_2^- &= 0.0375e^{-0.002\times400} = 0.01685 \quad [\text{V}], \\ I_2^- &= -0.0005e^{-0.002\times400} = -0.0002247 \quad [\text{A}] \\ V_2^+ &= 0.009375e^{-0.002\times400} = 0.0042125 \quad [\text{V}], \\ I_2^+ &= 0.000125e^{-0.002\times400} = 0.00005617 \quad [\text{A}] \end{aligned}$$

The total current and voltage at the generator is the sum of the forward- and backward-propagating waves for the duration of the narrow pulse (20 ns). The resulting voltage and current at the generator are shown in **Figure 16.8a**, which shows the voltage and current on the line at $t = 0$, $t = 2$ μs, $t = 4$ μs, etc. The values shown are the sums of the forward and backward amplitudes.

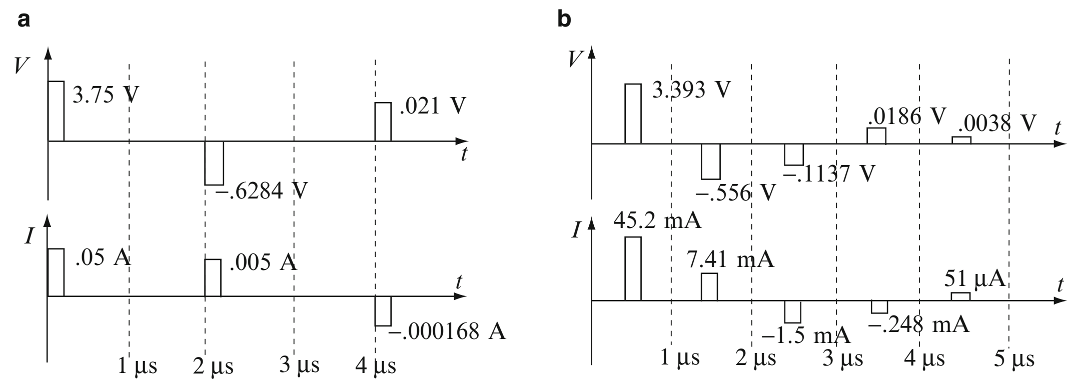


Figure 16.8 (**a**) Voltage and current at the generator in **Example 16.2**. (**b**) Voltage and current in the middle of the line in **Exercise 16.1**

Exercise 16.1 In **Example 16.2**, find the voltage and current in the middle of the transmission line for times $0 < t < 5$ μs.

Answer See **Figure 16.8b**.

Example 16.3 Application: Time Domain Reflectometry Time domain reflectometry (TDR) is a method of testing that relies on reflections from mismatched loads to locate the load. This is very useful in locating short circuits or cuts in inaccessible lines such as underground cables. A pulse is sent on the line and its reflections are recorded on a screen or chart. The distance between every two pulses is twice the time it takes to propagate to the fault. If the speed of propagation is known for the line, the exact location of the fault can be found. From the magnitude, shape, and sign of the signals, it is also possible to evaluate the type of fault (short, low, or high impedance, open) before repair. This can save considerable time and labor, especially if cables are buried.

A lossless underground telephone cable has inductance per unit length of 1 μH/m and capacitance of 25 pF/m. The cable has developed a fault and it is required to locate the fault and identify its nature. The time domain reflectometer reading looks as in **Figure 16.9b**:

(a) Find the distance of the fault from the source.
(b) What kind of fault does the cable have?

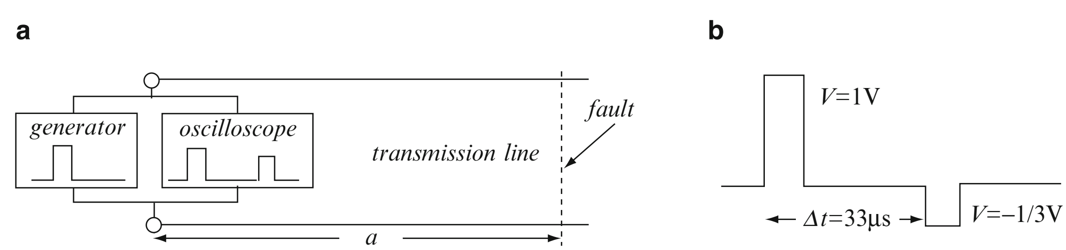


Figure 16.9 (**a**) A time domain reflectometer. (**b**) The signal obtained from the faulty cable

Solution: The distance to the fault is calculated from the time difference between two pulses and the speed of propagation on the line. The type of fault can be identified from the reflection coefficient at the fault:

(a) The speed of propagation on the line is

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times10^{-6}\times25\times10^{-12}}} = 2\times10^8 \quad \left[\frac{\text{m}}{\text{s}}\right].$$

The distance of the fault is

$$d = \frac{v_p\Delta t}{2} = \frac{2\times10^8\times3.3\times10^{-5}}{2} = 3,300 \quad [\text{m}].$$

(b) Because the first reflection is negative, the impedance at the load is smaller than the line impedance, as can be seen from the formula for the reflection coefficient at the load. The line impedance can be calculated from the inductance and capacitance per unit length:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{1\times10^{-6}}{25\times10^{-12}}} = 200 \quad [\Omega]$$

The reflection coefficient is

$$\Gamma_L = \frac{V^-}{V^+} = -\frac{1}{3} = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \rightarrow \quad Z_L = Z_0\frac{(1-1/3)}{(1+1/3)} = \frac{Z_0}{2} = 100 \quad [\Omega]$$

Thus, the fault is a “partial short,” such as may be caused by loss of insulation or water in the cable. The calculation of the fault impedance is only possible if the line is lossless and if the pulses do not distort. In practical applications, the line is never lossless and, therefore, the pulses are distorted. It is much more difficult to classify the fault exactly (although still possible), but the location of the fault is relatively easy to find. Also, step sources are often used and multiple reflection recorded to better analyze the fault.

Exercise 16.2 In **Example 16.3**, suppose that the amplitude of the reflected wave equals 99 % of the amplitude of the forward-propagating wave. What is the impedance of the fault if the intrinsic line impedance is $Z_0 = 200\ \Omega$?

Answer $Z_L = 39{,}800\ \Omega$. This is a partially open line.

16.4 Transients on Transmission Lines: Long Pulses

The condition considered here is that of a very long pulse, again, the length being related to the length of the line and speed of propagation. In other words, we assume now that $\Delta t \gg d/v_p$, where Δt is the pulse width, d the length of the line, and v_p the speed of propagation on the line. The main difference between this assumption and the assumption in the previous case is that the pulse can now propagate back and forth from generator to load during the pulse width Δt many times. In particular, a positively going or negatively going step function satisfies this condition. A number of pulses that may be considered here are shown in **Figure 16.10**.

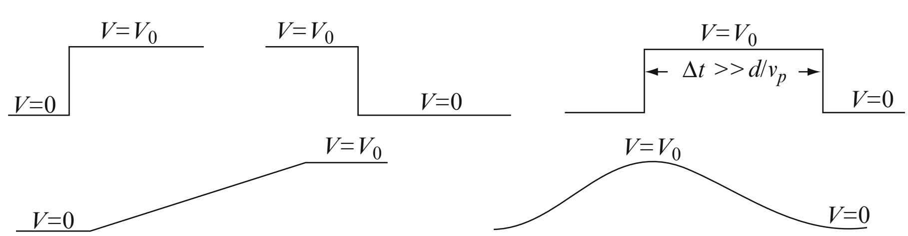


Figure 16.10 Some typical long pulses

Consider the circuit in **Figure 16.11a**. Initially, the switch is open and there is no current on the line. Suppose now the switch is closed at time $t = 0$. Initially, the condition is the same as in the previous case; that is, the disturbance on the line must propagate to the load starting at $t = 0$. The generator "sees" a load equal to Z_0 since no wave has propagated to the load yet. The voltage across the line and the current in the line at $z = 0$ are

$$V^+ = V_g \frac{Z_0}{Z_0 + Z_g} \quad [\mathrm{V}], \quad I^+ = \frac{V_g}{Z_0 + Z_g} \quad [\mathrm{A}] \tag{16.20}$$

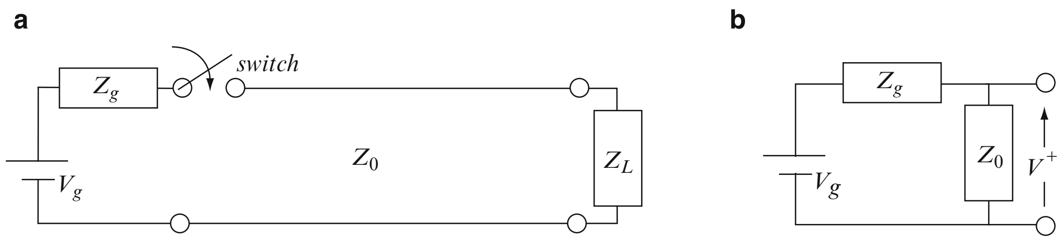
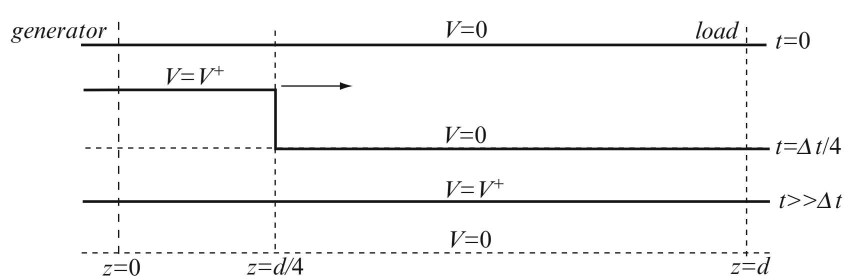


Figure 16.11 (**a**) A step pulse on a line generated by connecting the generator at $t = 0$. (**b**) Calculation of the forward waves V^+ and I^+ at the generator at $t = 0$

The equivalent circuit at $t = 0$ is shown in **Figure 16.11b** and is the same as a lumped parameter circuit. The closing of the switch has created a disturbance on the line: The forward wave V^+ now propagates toward the load at the speed of propagation v_p on the line. For a lossless or distortionless line, this speed is always given by **Eq. (16.1)** and is independent of the frequencies in the pulse. For a line of length d, the time of propagation to reach the end of the line is $\Delta t = d/v_p$. After this time, the forward-propagating wave appears at the load. There are three possible conditions that may occur at the load:

(1) Load impedance equals the characteristic impedance: $Z_L = Z_0$. In this case, the reflection coefficient at the load is zero. There is no reflection at the load and the circuit reaches steady state after a time $t = d/v_p$. The line voltage and line current are shown in **Figure 16.12** for three times.
(2) Load impedance greater than Z_0: $Z_L > Z_0$. In this case, the reflection coefficient is positive and, therefore, the reflected voltage wave is in the same direction as the forward-propagating wave. The reflected current at the load is in the direction opposite the forward current as shown in **Eq. (16.3)**.
(3) Load impedance less than Z_0: $Z_L < Z_0$. In this case, the reflection coefficient is negative ($\Gamma_L < 0$). The reflected voltage wave is opposite in polarity compared to the forward voltage wave, and the current is of the same polarity as the forward current wave.

Figure 16.12 Line voltage and current on a line with matched load, at different times and locations



Thus, we can treat cases 2 and 3 in identical fashion using the reflection coefficient, but in actual, numerical calculations, the sign of the reflection coefficient must be taken into account.

After the forward wave reaches the load, it is reflected. We call this the first reflection. The reflected waves are

$$V_1^- = \Gamma_L V^+ \quad [\mathrm{V}], \quad I_1^- = -\Gamma_L I^+ \quad [\mathrm{A}] \tag{16.21}$$

These two waves propagate back toward the generator as for the narrow pulse, but unlike the narrow pulse situation, the forward-propagating wave still exists on the line (since the pulse is very wide). Thus, the voltage (or current) anywhere on the line is the sum of the forward-propagating wave V^+ and backward-propagating wave V_1^- (I^+ and I_1^- for the current wave). The line voltage and current at any time $\Delta t < t < 2\Delta t$ are

$$V_1 = V^+(1+\Gamma_L) \quad [\mathrm{V}], \quad I_1 = I^+(1-\Gamma_L) \quad [\mathrm{A}], \quad \Delta t < t < 2\Delta t \tag{16.22}$$

After an additional time Δt ($2\Delta t$ from the time the switch was closed), the reflected wave V_1^- reaches the generator. Although the generator has its own voltage, it behaves as a load with load impedance Z_g for the reflected wave. Thus, a reflection coefficient Γ_g exists at the generator, unless $Z_g = Z_0$. For $Z_g \neq Z_0$, the generator reflection coefficient is given in **Eq. (16.9)**. Note that now the forward- and backward-propagating waves have changed roles. This should not be too confusing since the waves reflected from the load propagate backward toward the generator. These waves are reflected at the generator to produce new forward-propagating waves toward the load. These are

$$V_2^+ = \Gamma_g V_1^- = \Gamma_L \Gamma_g V^+ \quad [\mathrm{V}], \quad I_2^+ = -\Gamma_g I^- = \Gamma_L \Gamma_g I^+ \quad [\mathrm{A}] \tag{16.23}$$

The total voltage and current on the line at time $2\Delta t < t < 3\Delta t$ are

$$V_2 = V^+\left(1+\Gamma_L+\Gamma_L\Gamma_g\right) \quad [\mathrm{V}], \quad I_2 = I^+\left(1-\Gamma_L+\Gamma_L\Gamma_g\right) \quad [\mathrm{A}], \quad 2\Delta t < t < 3\Delta t \tag{16.24}$$

After an additional time Δt, the new forward-propagating waves (V_2^+ and I_2^+) reach the load and are reflected again. The new reflected waves, which then propagate backward toward the generator, are

$$V_3^- = \Gamma_L V_2^+ = \Gamma_L^2 \Gamma_g V^+ \quad [\mathrm{V}], \quad I_3^- = -\Gamma_L I_2^- = -\Gamma_L^2 \Gamma_g I^+ \quad [\mathrm{A}] \tag{16.25}$$

and the total line voltage and current are

$$V_3 = V^+\left(1+\Gamma_L+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g\right) \quad [\mathrm{V}], \quad I_3 = I^+\left(1-\Gamma_L+\Gamma_L\Gamma_g-\Gamma_L^2\Gamma_g\right) \quad [\mathrm{A}], \quad 3\Delta t < t < 4\Delta t \tag{16.26}$$

The pattern is now clear: Every reflection adds to (or subtracts from) the previous reflections to produce a total wave. Continuing the pattern, the voltage and current after many reflections may be written as

$$\begin{aligned} V &= V^+\left(1+\Gamma_L+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^2+\ldots\right) \\ &= V^+\left(1+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^3+\ldots\right) + V^+\Gamma_L\left(1+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^3+\ldots\right) \quad [\mathrm{V}] \end{aligned} \tag{16.27}$$

$$\begin{aligned} I &= I^+\left(1-\Gamma_L+\Gamma_L\Gamma_g-\Gamma_L^2\Gamma_g+\Gamma_L^2\Gamma_g^2-\Gamma_L^3\Gamma_g^2+\ldots\right) \\ &= I^+\left(1+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^3+\ldots\right) - I^+\Gamma_L\left(1+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^3+\ldots\right) \quad [\mathrm{A}] \end{aligned} \tag{16.28}$$

The term in parentheses is a geometric series (since $|\Gamma_L| < 1$, $|\Gamma_g| < 1$), and for a large number of terms, we get

$$1+\Gamma_L\Gamma_g+\Gamma_L^2\Gamma_g^2+\Gamma_L^3\Gamma_g^3+\ldots = \frac{1}{1-\Gamma_L\Gamma_g}, \quad |\Gamma_L|, |\Gamma_g| < 1 \tag{16.29}$$

Substituting in **Eq. (16.27)**, we get

$$\boxed{V_\infty = V^+\frac{1}{1-\Gamma_L\Gamma_g} + V^+\Gamma_L\frac{1}{1-\Gamma_L\Gamma_g} = V^+\frac{1+\Gamma_L}{1-\Gamma_L\Gamma_g} \quad [\mathrm{V}]} \tag{16.30}$$

Performing similar operations for I in **Eq. (16.28)**, we get

$$\boxed{I_\infty = I^+ \frac{1-\Gamma_L}{1-\Gamma_L\Gamma_g} \quad [\mathrm{A}]} \tag{16.31}$$

where the index indicates an infinite number of reflections (infinite time). This gives the steady-state solution for voltage and current on the line. Substituting for Γ_L and Γ_g from **Eqs. (16.3)** and **(16.9)**, and rearranging terms, we get

$$V_\infty = V^+ \frac{1+\Gamma_L}{1-\Gamma_L\Gamma_g} = V^+ \frac{Z_L(Z_0+Z_g)}{Z_0(Z_g+Z_L)} \quad [\mathrm{V}] \tag{16.32}$$

Now, substituting for V^+ from **Eq. (16.20)**, we get for the voltage on the line, which is also the voltage on the load at steady state,

$$\boxed{V_\infty = V_g \frac{Z_L}{Z_g+Z_L}} \quad [\mathrm{V}] \tag{16.33}$$

This is the steady-state solution for the circuit, as required. Similarly, for the current in the circuit (load), we get the steady-state solution as

$$\boxed{I_\infty = \frac{V_g}{Z_g+Z_L}} \quad [\mathrm{A}] \tag{16.34}$$

Although the method is simple and intuitive, it is rather lengthy, except for the steady-state solution. However, it is possible to reduce the method into a simple diagram which may be viewed as a tool for keeping track of the various reflections that occur. The diagram is called a reflection diagram (also called a bounce or Bewley diagram) and is shown in **Figures 16.13** through **16.15**. The method consists of the following:

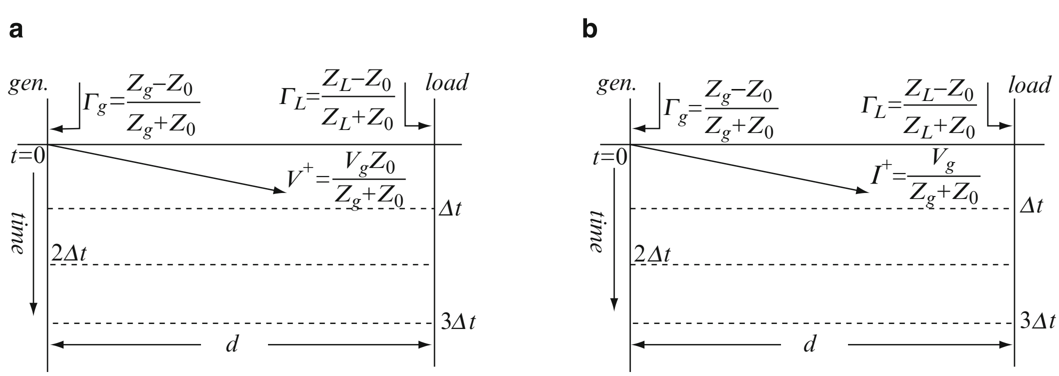


Figure 16.13 Preparatory steps in the reflection diagram. (**a**) Voltage reflection diagram. (**b**) Current reflection diagram

(1) The generator and load are replaced by two perpendicular lines separated a distance d apart. The horizontal distance represents location on the line, and the vertical axis represents time with $t = 0$, usually at the generator. The reflection coefficient at the generator (looking from the line into the generator) is placed on the left vertical line, whereas the reflection coefficient at the load (looking from the line into the load) is placed on the right vertical line. The same applies to the current diagram. These considerations are shown in **Figure 16.13**.

(2) Time is indicated along the lines starting from top to bottom in increments of $2\Delta t$. The left line is marked 0, $2\Delta t$, $4\Delta t$, $6\Delta t$, etc. The right line is marked Δt, $3\Delta t$, $5\Delta t$, $7\Delta t$, etc. This conforms with the above notation and indicates that a wave propagates between generator and load or vice versa in a time $\Delta t = d/v_p$.

(3) The initial voltage and current, at time $t = 0$, are calculated from **Eq. (16.20)**. These are marked at time $t = 0$ on the diagram, pointing toward the load as shown in **Figure 16.13**.

(4) The foregoing steps give the initial or preparatory steps. Now, we allow the initial waves to propagate, and each encounter with a reflection coefficient multiplies the wave by that reflection coefficient **[Eq. (16.21)]** and changes the direction of propagation. **Figure 16.14** shows a few steps in the diagram. All odd-numbered reflections occur at the load; all even-numbered reflections occur at the generator.

(5) To calculate the voltage or current at any point on the line and at any time, we proceed by marking the location at which the values are required. For example, suppose we wish to calculate the line voltage and line current at point z_0 in **Figure 16.14**. A line parallel to the load or generator line is drawn at $z = z_0$. This line shows the voltage at any point in time from zero (top) to infinity (bottom). The line $z = z_0$ intersects the reflected voltages and currents at times t_1, t_2, t_3, etc., as shown. The line voltage and current are shown in **Figure 16.15**. Note that in this figure, both Γ_L and Γ_g are assumed to be positive. Thus, the voltage at z_0 increases in diminishing steps. The values of voltage or current remain constant between two reflections, until an additional reflection reaches the same point.

(6) The voltage or current at any given time at a given point between generator and load is calculated by summing up all reflections for all times up to the required time, at the required point. As an example, the voltage and current at time $t = t_0$ at $z = z_0$ in **Figure 16.14** is the sum of the first four reflections and the initial voltage. In this case,

$$V_0 = V^+\left(1 + \Gamma_L + \Gamma_L\Gamma_g + \Gamma_L^2\Gamma_g + \Gamma_L^2\Gamma_g^2\right) \quad [\mathrm{V}], \quad I_0 = I^+\left(1 - \Gamma_L + \Gamma_L\Gamma_g - \Gamma_L^2\Gamma_g + \Gamma_L^2\Gamma_g^2\right) \quad [\mathrm{A}] \qquad (16.35)$$

These values are shown in **Figure 16.15**.

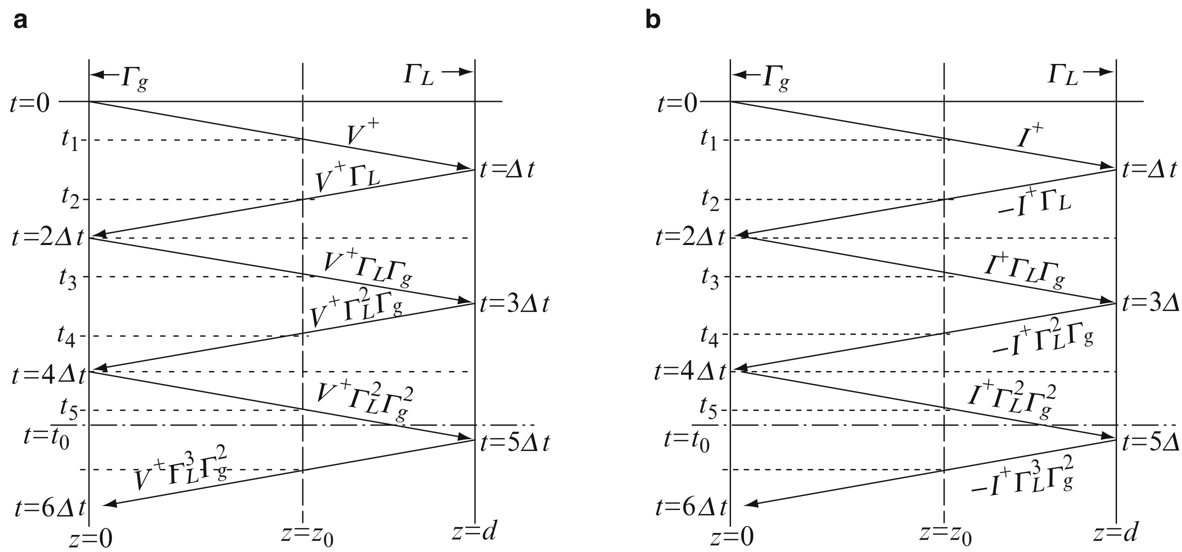
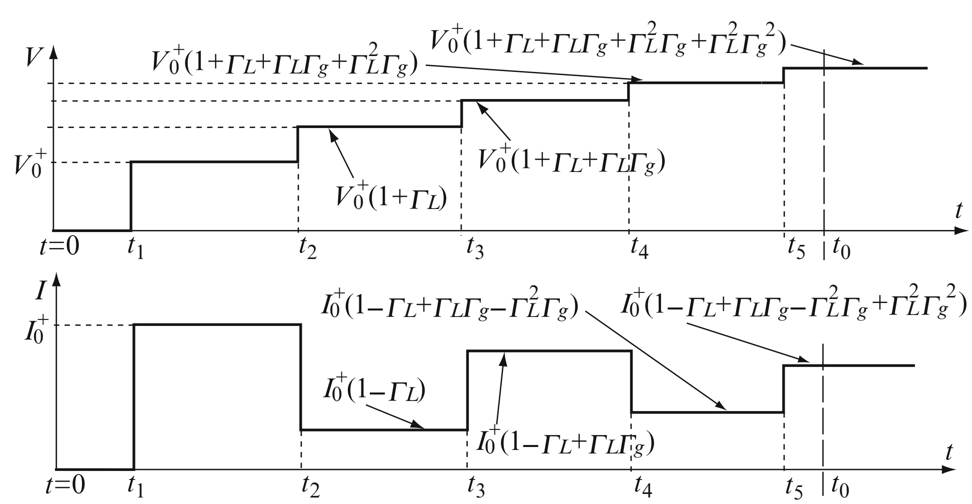


Figure 16.14 (**a**) The voltage reflection diagram for a general transmission line with reflection coefficients Γ_L and Γ_g. (**b**) The current reflection diagram for the conditions in (**a**)

Figure 16.15 Voltage and current on the line at a given location as a function of time



Example 16.4 A transmission line is connected as shown in **Figure 16.16**. The inductance per unit length of the line is 5 μH/m, and the capacitance per unit length is 5 pF/m. The switch is closed at $t = 0$. Calculate:

(a) The steady-state voltage and current on the line.
(b) The voltage at the load as measured by an oscilloscope between $t = 0$ and $t = 3$ μs.
(c) The current midway between generator and load as measured between $t = 0$ and $t = 3$ μs.

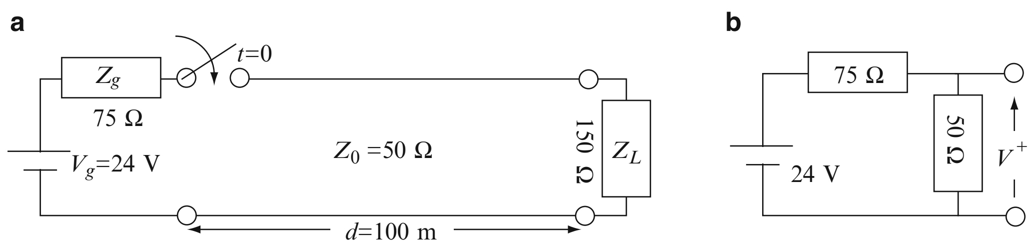


Figure 16.16 A transmission line on which the generator is switched on at $t = 0$

Solution: For steady state, we can either use **Eqs. (16.30)** and **(16.31)** or **Eqs. (16.33)** and **(16.34)**. The former will be used here. As for the transient solution, we use **Eqs. (16.27)** and **(16.28)** with the appropriate number of reflections. The latter is found from the length of the line and speed of propagation:

(a) The speed of propagation on the line is $v_p = 1/\sqrt{LC} = 2 \times 10^8$ m/s. Thus, the time required for propagation between the generator and load is 0.5 μs. To calculate the steady-state solution and to build the reflection diagram, we need the reflection coefficients at the load and generator (looking into the load or generator, respectively) and the initial voltage and current at $t = 0$ (V^+ and I^+). These are

$$\Gamma_L \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{150 - 50}{150 + 50} = 0.5, \quad \Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{75 - 50}{75 + 50} = 0.2$$

$$V^+ = \frac{V_0 Z_0}{Z_g + Z_0} = \frac{24 \times 50}{125} = 9.6 \quad [\text{V}], \quad I^+ = \frac{V_0}{Z_g + Z_0} = \frac{24}{125} = 0.192 \quad [\text{A}]$$

The steady-state solution is

$$V_\infty = V^+ \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_g} = 9.6 \times \frac{1 + 0.5}{1 - 0.5 \times 0.2} = 16 \quad [\text{V}],$$

$$I_\infty = I^+ \frac{1 - \Gamma_L}{1 - \Gamma_L \Gamma_g} = 0.192 \times \frac{0.5}{0.9} = 0.1067 \quad [\text{A}].$$

(b) The reflection diagram for voltages is now as in **Figure 16.17a**, where the first few reflections are shown. The time $t = 3$ μs is shown as a horizontal line. The voltage at the load is the sum of all values at the load from $t = 0$ to $t = 3$ μs since all remain on the line indefinitely (the pulse is very long). These are shown in **Figure 16.17b**. Note the way the diagram is drawn in comparison to **Figure 16.15**. The steady state in this case is reached quite fast. At $t = 3$ μs, the load voltage is 15.984 V which is only 16 mV lower than the steady state voltage.

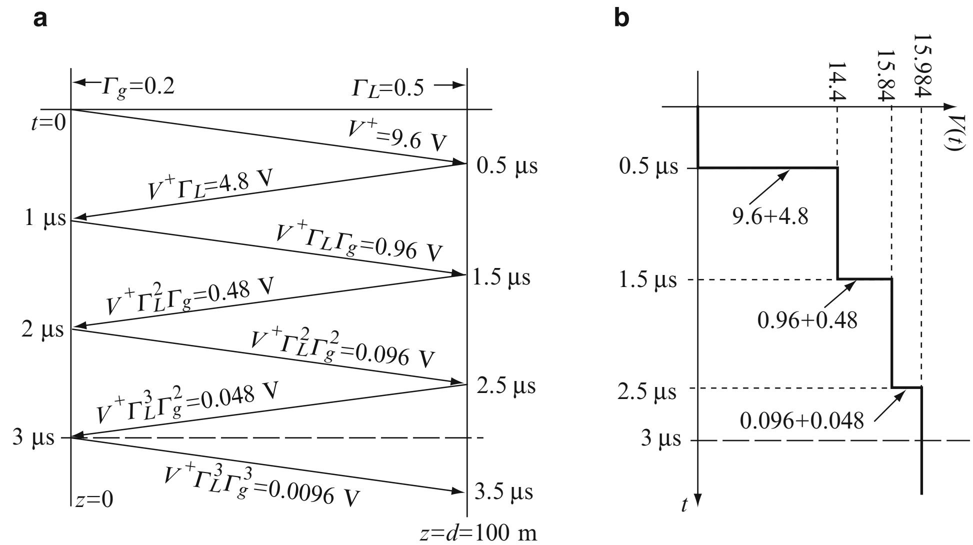


Figure 16.17 (**a**) Voltage reflection diagram for the line in **Figure 16.16**. (**b**) Voltage at the load in **Figure 16.16**

(c) The current midway between generator and load is found from the current reflection diagram in **Figure 16.18a**. The horizontal line at $t = 3$ μs and the vertical line at $z = d/2$ are shown. The plot of current with time is shown in **Figure 16.18b**. Note that the current is zero for the first 0.25 μs. Then, it remains constant for 0.5 μs until the reflected wave reaches this point again, and so on. The current at $t = 3$ μs is 0.10656 A. The line is almost at steady state.

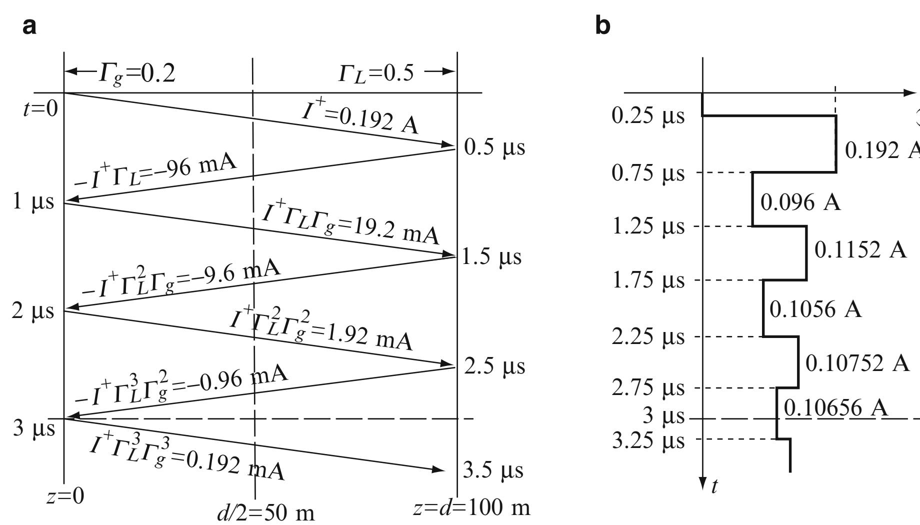


Figure 16.18 (**a**) Current reflection diagram for **Figure 16.16**. (**b**) Current midway between load and generator in **Figure 16.16**

16.5 Transients on Transmission Lines: Finite-Length Pulses

In the preceding two sections, we discussed the behavior of two types of pulses. One was a very short pulse and the second was very long. If, instead, a finite-width pulse is prescribed, we can use the superposition of solutions we already obtained to calculate the transmission line response to the pulse. A method of obtaining a pulse of width T is shown in **Figure 16.19**. In essence, we create a finite duration pulse as a superposition of two step functions. The first step function is applied at a time $t = 0$ and the second is applied at a time $t + T$. This, of course, is done so that we may use the solution in the previous section. Each step function is evaluated separately, and then the results are added to obtain the pulse response. The additional important point is to displace the second diagram by a time T to ensure that the correct pulse width is created. This method can be extended to almost any pulse shape, although the method may be lengthy. For example, a triangular pulse may be approximated by any number of steps. If the steps are small and a large number of steps are used, the pulse may be approximated quite accurately. The approximation for a triangular pulse is shown in **Figure 16.20**, using four steps on the rising edge and four steps on the falling edge. The first four pulses are exactly the same, but the first pulse starts at $t_0 + T/16$ and each subsequent pulse is displaced an additional $T/8$. The net effect is a narrowing of the pulse compared to the actual triangular pulse, but this is of minor concern since we can decrease this narrowing by increasing the number of pulses we use. The last four pulses are the same in magnitude but are negative. The following example shows how this method is applied.

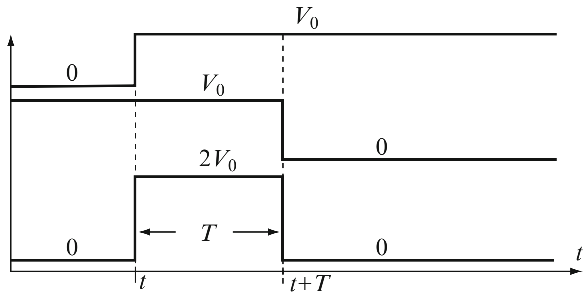
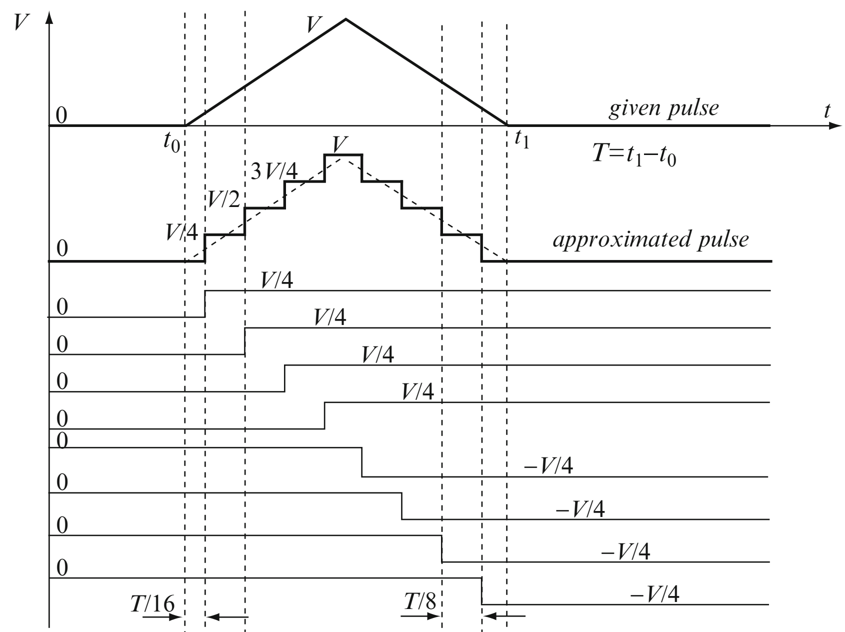


Figure 16.19 The superposition of two shifted step pulses results in a finite duration pulse

Figure 16.20 Approximation of a triangular pulse by step pulses



Example 16.5 Transient Due to a Triangular Pulse The transmission line in **Figure 16.21a** is driven with a single triangular pulse as shown. The speed of propagation on the line is 10^8 m/s:

(a) Find the current in the load at all times between $t = 0$ and $t = 50$ μs.
(b) Find the steady-state voltage on the line.

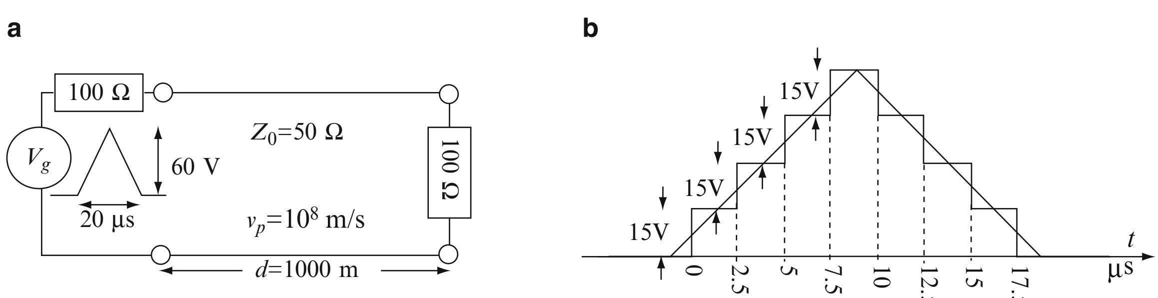


Figure 16.21 (**a**) A transmission line driven by a single triangular pulse. (**b**) Representation of the triangular pulse as a combination of steps

Solution: To solve the problem, we can divide the pulse into any number of steps. The larger the number of steps, the better the approximation to the exact solution. Here, we choose to divide the pulse into four steps on each slope, as in **Figure 16.21b**. The solution is then a superposition of four positive steps and four negative steps, each of magnitude 15 V/150 Ω = 0.1 A. The reflection diagram for one positive (or negative) step is shown in **Figure 16.22a**. The reflection coefficients are shown on the diagram:

(a) The solution involves some approximations. The most obvious is the use of the finite number of steps. The second approximation necessary is shown in **Figure 16.21b**. The pulses are chosen such that they approximate the original triangular pulse which passes through the centers of the vertical and horizontal lines forming the pulse. The width of the approximate pulse is only 17.5 μs with each pulse displaced 2.5 μs with respect to the other. Also, the first pulse starts 1.25 μs from the time the true triangular pulse starts, but, in the interest of simplicity, we start the first pulse at $t = 0$. From the diagram in **Figure 16.22a**, the current in the load is calculated and shown in **Figure 16.22b** for the first step. Note that the first jump occurs at $t = 10$ μs and is equal to $0.1 - 0.1/3 = 0.0667$ A. The second jump at $t = 30$ μs adds $0.1/9 - 0.1/27 = 0.0074$ A. The remaining three pulses are the same, but are displaced to the right by 2.5 μs each. Similarly, the negative pulses are identical in form but negative, and they are also displaced by 2.5 μs each with respect to the previous pulse. If we draw the eight pulses with the proper shift in time, we get the result in **Figure 16.23**. The result is the sum of all eight pulses and is shown at the bottom of the diagram. Note, in particular, the multiple pulses produced by the multiple reflections. These pulses die out with time.

(b) The steady-state voltage on the line is zero. This can be seen from **Figure 16.23**. The steady-state response to each step is identical except for signs. There are four positive responses and four negative responses. Their sum is zero; that is, the pulse is eventually dissipated.

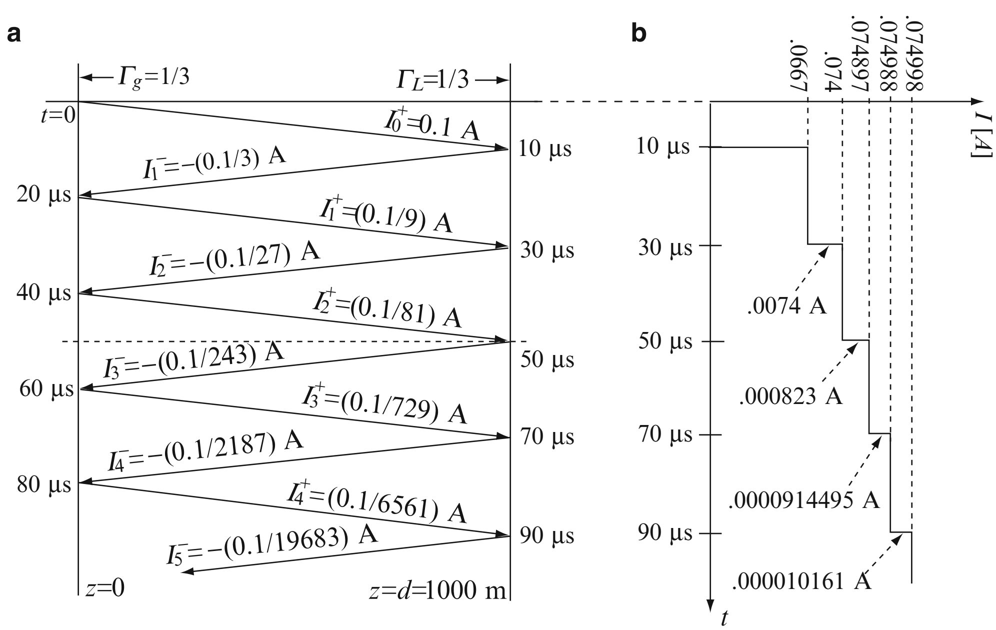


Figure 16.22 (**a**) Current reflection diagram for the first step in **Figure 16.21b**. (**b**) Current at the load due to the first pulse in **Figure 16.21b**

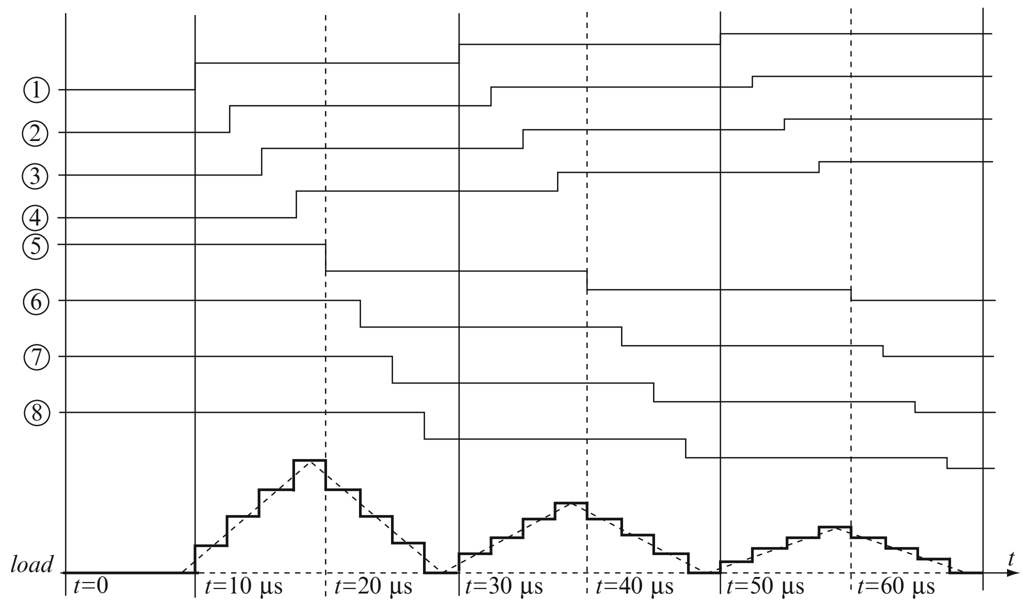


Figure 16.23 Superposition of the responses of the eight pulses that make up the triangular signal

16.6 Reflections from Discontinuities

A discontinuity on a line is any condition that changes the impedance on the line. For example, the connecting point between two lines of different characteristic impedances is a discontinuity that will cause a reflection at the point of discontinuity. Similarly, a uniform line on which a load has been connected somewhere on the line becomes a discontinuous line. These two situations are shown in **Figures 16.24a** and **16.24b**. A similar situation is caused by connecting more than one transmission line at the end of a transmission line as shown in **Figure 16.24c**. The introduction of a discontinuity causes both reflections and transmission of waves at the discontinuity as well as at any other location at which there is a mismatch in impedance. To understand the behavior of the transient waves in the presence of a discontinuity, consider **Figure 16.24a**. The waves are found as for the mismatched load in **Section 16.4**, but now we have three locations to deal with: load, generator, and discontinuity. If there is more than one discontinuity, each discontinuity must be treated separately.

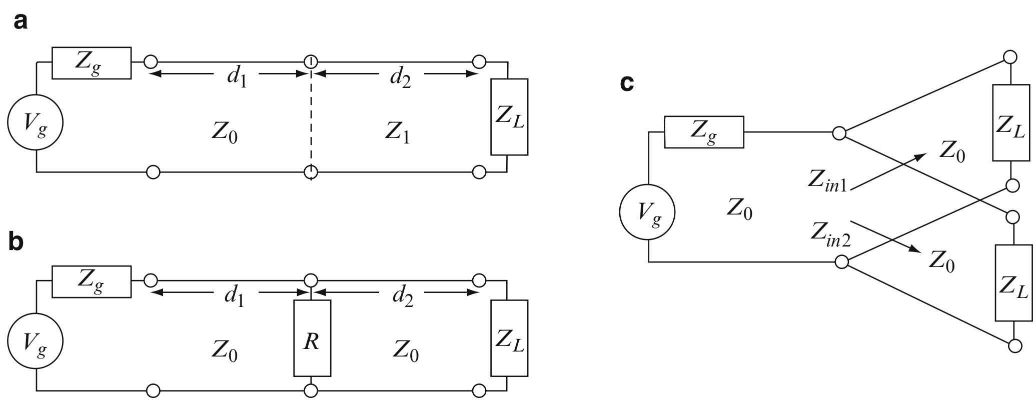


Figure 16.24 Discontinues on transmission lines. (**a**) Due to connection of two lines. (**b**) Due to connection of loads on the lines. (**c**) Due to a distribution point

To understand how the waves behave, we will follow the propagation of waves in **Figure 16.24a** and draw the reflection diagram as we go along. For simplicity, we assume that the generator is matched ($Z_g = Z_0$). Therefore, the forward-propagating wave launched by the generator at time $t = 0$ is

$$V_0^+ = \frac{V_g Z_0}{Z_0 + Z_0} = \frac{V_g}{2} \quad [\mathrm{V}] \tag{16.36}$$

This wave propagates on line 1 at a speed of propagation v_{p1}. After a time $\Delta t_1 = d_1/v_{p1}$, the wave reaches the discontinuity. Part of the wave is reflected and part of it is transmitted with the reflection and transmission coefficients Γ_{12} and T_{12}, respectively:

$$\Gamma_{12} = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \quad T_{12} = \frac{2Z_1}{Z_1 + Z_0} \tag{16.37}$$

The reflection coefficient Γ_{12} is the reflection coefficient at the interface between line 1 and line 2, and the transmission coefficient indicates the transmission from line 1 to line 2. These two coefficients are shown in **Figure 16.25**, where the arrows indicate the direction of the waves being reflected and transmitted. The reflected and transmitted voltage waves at d_1 are

$$V_1^- = V_0^+ \Gamma_{12}, \qquad V_1^+ = V_0^+ T_{12} \quad [\mathrm{V}] \tag{16.38}$$

The reflected wave V_1^- propagates back to the generator and reaches the generator after a time Δt_1. Since the reflection coefficient at the generator is zero, no additional reflections occur at this point. The wave transmitted across the discontinuity, V_1^+, propagates toward the load at a speed of propagation v_{p2} and reaches the load after an additional time $\Delta t_2 = d_2/v_{p2}$. At the load, the wave is partly reflected and partly transmitted into the load (where it is dissipated or, in the case of an antenna, radiated). The reflection and transmission coefficients at the load are

$$\Gamma_L = \frac{Z_L - Z_1}{Z_L + Z_1}, \quad T_L = \frac{2Z_L}{Z_L + Z_1} \tag{16.39}$$

Thus, the reflected and transmitted waves are

$$V_2^- = V_2^+ \Gamma_L = V_0^+ T_{12} \Gamma_L, \qquad V_{L1}^+ = V_0^+ T_{12} T_L \quad [\mathrm{V}] \tag{16.40}$$

V_2^- propagates back toward the discontinuity, which it reaches after an additional time Δt_2. At the discontinuity, there will be a reflected and transmitted wave, but since the wave reaches the discontinuity from line 2, the reflection and transmission coefficients are different. These are denoted Γ_{21} and T_{21}:

$$\Gamma_{21} = \frac{Z_0 - Z_1}{Z_1 + Z_0}, \quad T_{21} = \frac{2Z_0}{Z_1 + Z_0} \tag{16.41}$$

The reflected wave (into line 2) and the transmitted wave (from line 2 into line 1) are

$$V_3^+ = V_2^- \Gamma_{21} = V_0^+ T_{12} \Gamma_L \Gamma_{21}, \qquad V_3^- = V_0^+ T_{12} \Gamma_L T_{21} \quad [\mathrm{V}] \tag{16.42}$$

Now, these two waves propagate in opposite directions. V_3^+ propagates toward the load whereas V_3^- propagates toward the generator. The sequence repeats itself indefinitely. A few reflections are shown in **Figure 16.25**, together with the definitions of reflection and transmission coefficients at the various locations.

All other aspects of propagation remain as discussed in **Section 16.4**. Note, in particular, the times at which the waves reach various locations on the line. The main difficulty in treating discontinuities is in keeping track of the increasing number of reflections and transmissions and the associated times. We note also that the reflection and transmission coefficients at the discontinuity depend on the direction of propagation. The following relations hold:

$$\Gamma_{21} = -\Gamma_{12}, \quad T_{21} = 1 - \Gamma_{12} \tag{16.43}$$

and these can be obtained from **Eqs. (16.37)** and **(16.41)**. Once the diagrams are defined, the waves at any location on the line may be found as previously, by finding the intersection of the time line and position line (t_0 and z_0 in **Figure 16.25**) and summing all terms up to that time along the time line. These aspects of calculation are demonstrated in **Example 16.6**. Clearly, an essentially identical process applies to the current diagram.

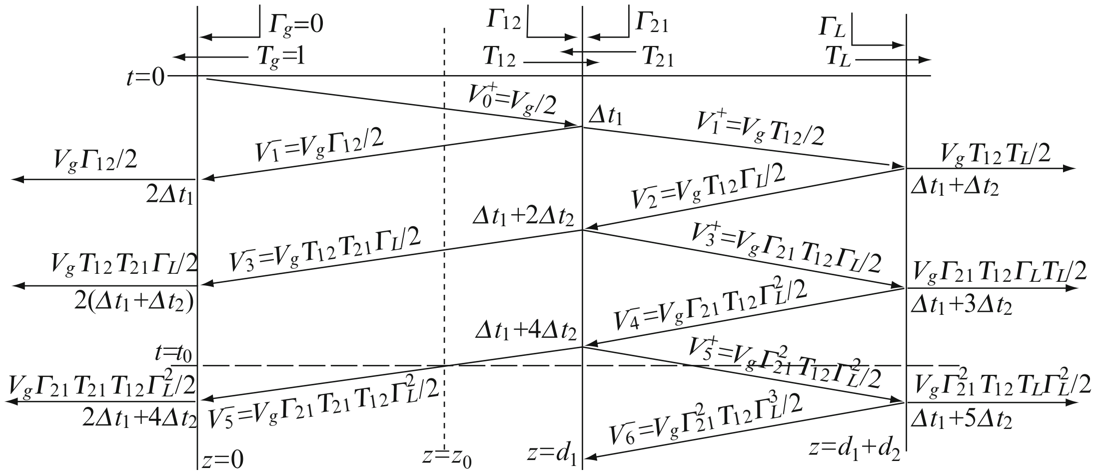


Figure 16.25 Voltage reflection diagram for the line in **Figure 16.24a**, with $Z_g = Z_0$

Example 16.6 Application: Line Patching A segment of a lossless transmission line of finite length $d = 100$ m and characteristic impedance $Z_2 = 75\ \Omega$ is connected between two infinite lossless lines, each with characteristic impedance $Z_1 = Z_3 = 50\ \Omega$ as a temporary fix until the proper line can be obtained, as shown in **Figure 16.26**. A step voltage V_0 arrives at the connection between lines 1 and 2 at $t = 0$ from the left. The speed of propagation on the lines is $v_p = 10^8$ m/s. With the properties given in the figure, calculate the voltage on each line at $t = 5.8$ μs. In lines 1 and 3, calculate the voltage at the discontinuity. In line 2, calculate it midway.

Solution: In the two infinite lines, there can be no reflections except at the two connections shown. At the discontinuities there are two reflection coefficients and two transmission coefficients as shown in **Figures 16.26** and **16.27**. The latter figure also shows the first few reflected and transmitted waves at both discontinuities. These are the only waves possible. To find the wave on each line at a given time, the time and position lines are drawn, shown as dashed lines in **Figure 16.27**, and the terms up to the given time and position are summed up.

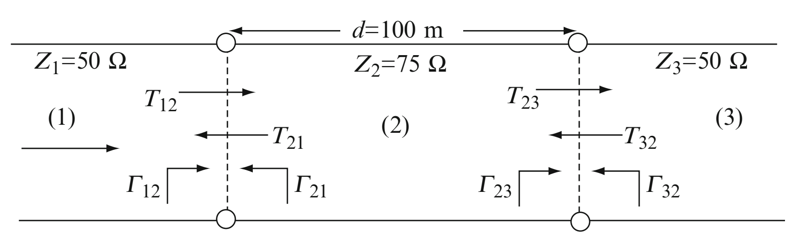


Figure 16.26 A finite transmission line segment connected between two infinite lines. The various reflection and transmission coefficients are shown

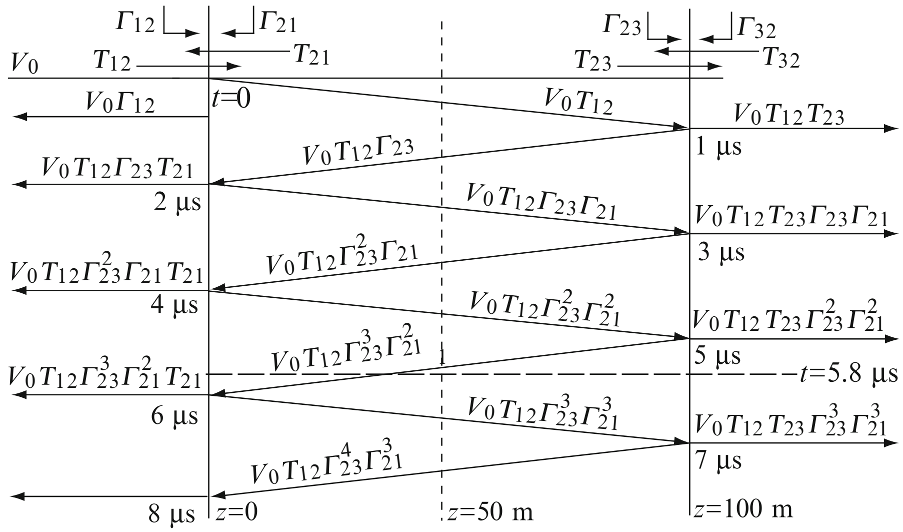


Figure 16.27 Voltage reflection diagram for the line in **Figure 16.26**

On line 1 (immediately to the left of the connection ($z = 0^-$)):

$$V_{t=5.8s} = V_0[1 + \Gamma_{12} + T_{12}T_{21}\Gamma_{23}(1 + \Gamma_{23}\Gamma_{21})] \quad [\text{V}]$$

On line 2 (at the center of the line ($z = 50$ m)):

$$V_{t=5.8s} = V_0 T_{12}\left[1 + \Gamma_{23} + \Gamma_{23}\Gamma_{21} + \Gamma_{23}^2\Gamma_{21} + \Gamma_{23}^2\Gamma_{21}^2 + \Gamma_{23}^3\Gamma_{21}^2\right] \quad [\text{V}]$$

On line 3 (immediately to the right of the connection ($z = 100$ m$^+$)):

$$V_{t=5.8s} = V_0 T_{23} T_{12}\left[1 + \Gamma_{23}\Gamma_{21} + \Gamma_{23}^2\Gamma_{21}^2)\right] \quad [\text{V}]$$

The various reflection and transmission coefficients needed are

$$\Gamma_{12} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{75 - 50}{75 + 50} = 0.2, \qquad T_{12} = \frac{2Z_2}{Z_2 + Z_1} = \frac{150}{125} = 1.2,$$

$$\Gamma_{21} = -\Gamma_{12} = -0.2, \qquad T_{21} = \frac{2Z_1}{Z_2 + Z_1} = \frac{100}{125} = 0.8,$$

$$\Gamma_{23} = \frac{Z_3 - Z_2}{Z_3 + Z_2} = \frac{50 - 75}{50 + 75} = -0.2, \qquad T_{23} = \frac{2Z_3}{Z_3 + Z_2} = \frac{100}{125} = 0.8,$$

The voltages are as follows:
In line 1, immediately to the left of the discontinuity:

$$V_{t=5.8s} = V_0[1 + 0.2 - 1.2 \times 0.8 \times 0.2 \times (1 + 0.04)] = 1.00032V_0 \quad [\text{V}]$$

In line 2, at the center of the line:

$$V_{t=5.8s} = V_0 \times 1.2 \times [1 - 0.2 + 0.2 \times 0.2 - 0.04 \times 0.2 + 0.04 \times 0.04 - 0.008 \times 0.04] = 0.999936V_0 \quad [\text{V}]$$

In line 3, immediately to the right of the discontinuity:

$$V_{t=5.8s} = V_0 \times 0.8 \times 1.2 \times [1 + 0.2 \times 0.2 + 0.04 \times 0.04] = 0.999936V_0 \quad [\text{V}]$$

Exercise 16.3

(a) Calculate the steady-state voltage on the three lines in **Example 16.6** using the general coefficients.
(b) With the constants found in **Example 16.6**, show that the steady-state voltages are equal to V_0.

Answer

(a)

$$V_1 = V_0\left[1 + \Gamma_{12} + \frac{\Gamma_{23}T_{12}T_{21}}{1 - \Gamma_{23}\Gamma_{21}}\right], \quad V_2 = V_0 T_{12}\frac{1+\Gamma_{23}}{1-\Gamma_{23}\Gamma_{21}}, \quad V_3 = V_0\frac{T_{23}T_{12}}{1-\Gamma_{23}\Gamma_{21}} \quad [\text{V}].$$

16.7 Transients on Lines with Reactive Loading

The transient representation in the previous section was based on the concept of reflection and the reflection coefficient. The reflection coefficient is only properly defined if the reflected wave is directly proportional to the forward-propagating wave. In other words, to calculate the reflection coefficient, we assumed that $V^- = \Gamma V^+$. If, however, the reflected wave depends on the forward wave's amplitude in a nonlinear fashion, then the reflection coefficient is not a constant and the method of the previous sections cannot be used. As an example, suppose that a line is terminated with a nonlinear resistor, whose resistance depends on the line voltage as

$$Z_L = R_0\left(1 + kV^2\right) \qquad [\Omega] \tag{16.44}$$

where V is the total voltage on the load. Assuming the characteristic impedance of the line is $Z_0 = R_0$, the reflection coefficient is

$$\Gamma_L = \frac{R_0\left(1 + kV^2\right) - R_0}{R_0\left(1 + kV^2\right) + R_0} = \frac{kV^2}{kV^2 + 2} \tag{16.45}$$

This reflection coefficient cannot be used in the relations in **Sections 16.2** through **16.6** because it is not a constant. Thus, we must resort to other means when trying to find the transients on the line. Note that if we had a method of evaluating the voltage in **Eq. (16.45)**, then Γ_L could be evaluated and the methods of the previous section would apply. Thus, the basic method is to calculate the forward-propagating wave and, from this, to calculate the reflected wave without resorting to the use of the reflection coefficient. To see how this is done, we consider two situations: the first deals with capacitive loading and the second with inductive loading.

16.7.1 Capacitive Loading

Consider a transmission line with characteristic impedance $Z_0 = R_0$ connected to a generator with internal impedance $Z_g = R_g$ and a capacitor as a load as shown in **Figure 16.28**.

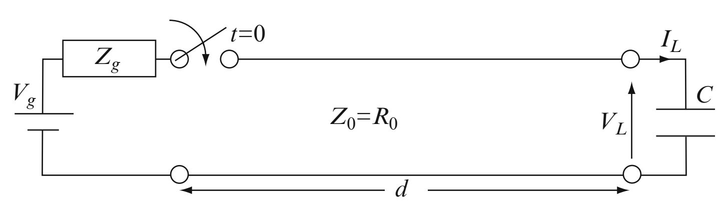


Figure 16.28 A capacitively loaded transmission line

The calculation starts by calculating the forward-propagating wave, as in **Eq. (16.20)**. The initial voltage and current on the line (immediately after closing the switch) are

$$V^{+} = V_g \frac{R_0}{R_0 + R_g}, \quad I^{+} = \frac{V_g}{R_0 + R_g} \quad [\mathrm{V}] \tag{16.46}$$

These waves propagate toward the load at a speed v_p defined by the line parameters. At the load, however, the reflected voltage and current must be calculated from the differential equation relating current and voltage for a capacitor, because a reflection coefficient based on impedances cannot be used:

$$i_L(t) = C\frac{d}{dt}(v_L(t)) \quad [\mathrm{A}] \tag{16.47}$$

where $v_L(t)$ is the total voltage at the load. Note also that this voltage is time dependent, whereas V^+ is a constant voltage, and that $i_L(t)$ only exists after a time $t \geq \Delta t$. We can also write at the load the general relations

$$v_L(t) = V^{+} + V^{-}(t) \quad [\mathrm{V}], \quad i_L(t) = \frac{V^{+} - V^{-}(t)}{R_0} \quad [\mathrm{A}] \tag{16.48}$$

Solving for $i_L(t)$,

$$i_L(t) = \frac{2V^{+} - v_L(t)}{R_0} \quad [\mathrm{A}] \tag{16.49}$$

Substituting this in **Eq. (16.47)** and rearranging terms gives

$$C\frac{d}{dt}(v_L(t)) + \frac{1}{R_0}v_L(t) - \frac{2V^{+}}{R_0} = 0 \tag{16.50}$$

Since V^+ is known from **Eq. (16.46)**, we can solve this differential equation for any time $t \geq \Delta t$. The solution gives the voltage at the load:

$$\boxed{v_L(t) = 2V^{+}\left(1 - e^{-(t-\Delta t)/R_0C}\right) = \frac{2V_gR_0}{R_0 + R_g}\left(1 - e^{-(t-\Delta t)/R_0C}\right) \quad [\mathrm{V}], \quad t \geq \Delta t} \tag{16.51}$$

The current in the load is

$$\boxed{i_L(t) = \frac{2V^{+} - v_L(t)}{R_0} = \frac{2V^{+}e^{-(t-\Delta t)/R_0C}}{R_0} = \frac{2V_gR_0e^{-(t-\Delta t)/R_0C}}{R_0(R_0 + R_g)} \quad [\mathrm{A}], \quad t \geq \Delta t} \tag{16.52}$$

Now, the reflected voltage and current waves can be calculated from **Eq. (16.48)**:

$$V_1^{-}(t) = V^{+}\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) = \frac{V_gR_0}{R_0 + R_g}\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) \quad [\mathrm{V}] \tag{16.53}$$

$$I_1^{-}(t) = -\frac{V^{-}(t)}{R_0} = -\frac{V^{+}}{R_0}\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) = -\frac{V_gR_0}{R_0(R_0 + R_g)}\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) \quad [\mathrm{A}] \tag{16.54}$$

The total voltage and current on the line are given by the sum of the forward- and backward-propagating waves. The forward, reflected, and total voltages on the line are shown in **Figure 16.29a**. The load voltage and current are shown in **Figure 16.29b**.

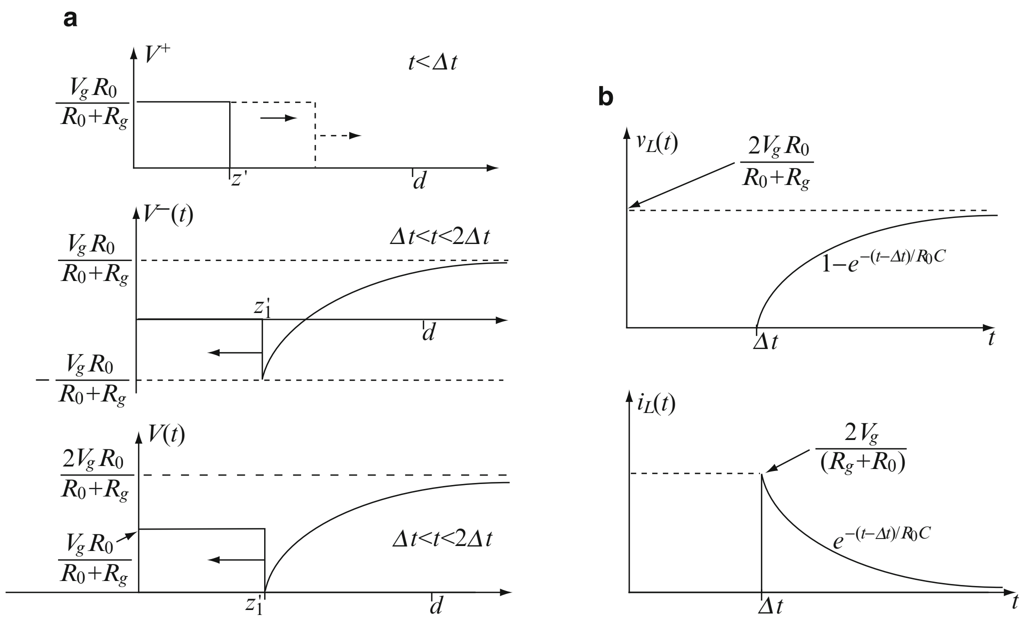


Figure 16.29 (**a**) Forward, reflected, and total voltage on the line in **Figure 16.28**. (**b**) Load voltage and current for the line in **Figure 16.28**

The backward-propagating waves in **Eqs. (16.53)** and **(16.54)** propagate toward the generator. If the generator is matched to the line (i.e., $R_g = R_0$), there will be no reflection at the generator and the solutions in **Eqs. (16.51)** and **(16.52)** apply. If, on the other hand, the generator is not matched, there will be a reflection at the generator as well, but because the generator impedance is resistive, the reflection coefficient Γ_g can be used as in the previous cases. A new forward-propagating wave is obtained which again travels toward the load, and the above steps are repeated. Any number of reflections may be considered in this way, and a steady state is achieved only after a large (infinite) number of reflections have occurred.

16.7.2 Inductive Loading

If an inductor replaces the capacitor in **Figure 16.28**, the treatment is similar except that now the basic equation to deal with is

$$v_L(t) = L\frac{d}{dt}(i_L(t)) \quad [\mathrm{V}] \tag{16.55}$$

All other aspects, including the relations at the load **[Eqs. (16.48)** and **(16.49)]** and the forward-propagating wave, are the same as for the capacitive load.

The differential equation to solve at the load is now

$$L\frac{d}{dt}(i_L(t)) + R_0 i_L(t) - 2V^+ = 0 \tag{16.56}$$

This gives the current at the load as

$$\boxed{i_L(t) = \frac{2V^+}{R_0}\left(1 - e^{-(t-\Delta t)R_0/L}\right) = \frac{2V_g R_0}{R_0(R_0+R_g)}\left(1 - e^{-(t-\Delta t)R_0/L}\right) \quad [\mathrm{A}], \quad t \geq \Delta t} \tag{16.57}$$

and the voltage as

$$\boxed{v_L(t) = 2V^+ e^{-(t-\Delta t)R_0/L} = \frac{2V_g R_0}{R_0+R_g} e^{-(t-\Delta t)R_0/L} \quad [\mathrm{V}], \quad t \geq \Delta t} \tag{16.58}$$

The reflected voltage and current are

$$V_1^-(t) = v_L(t) - V^+ = V^+\left(2e^{-(t-\Delta t)R_0/L} - 1\right) = \frac{V_g R_0}{R_0 + R_g}\left(2e^{-(t-\Delta t)R_0/L} - 1\right) \quad [\text{V}] \tag{16.59}$$

$$I_1^-(t) = -\frac{v_L(t) - V^+}{R_0} = -\frac{V_g R_0}{R_0\left(R_0 + R_g\right)}\left(2e^{-(t-\Delta t)R_0/L} - 1\right) \quad [\text{A}] \tag{16.60}$$

Figure 16.30a shows these relations and their variation on the line and with time, and **Figure 16.30b** shows the voltage and current at the load. As was the case with the capacitive loading in the previous section, if the generator is matched, the results here describe the behavior of the line at all times. If the generator is not matched, the above behavior only applies up to a time $t = 2\Delta t$. At this time, the backward-propagating wave reaches the generator and is reflected, generating a new forward-propagating wave.

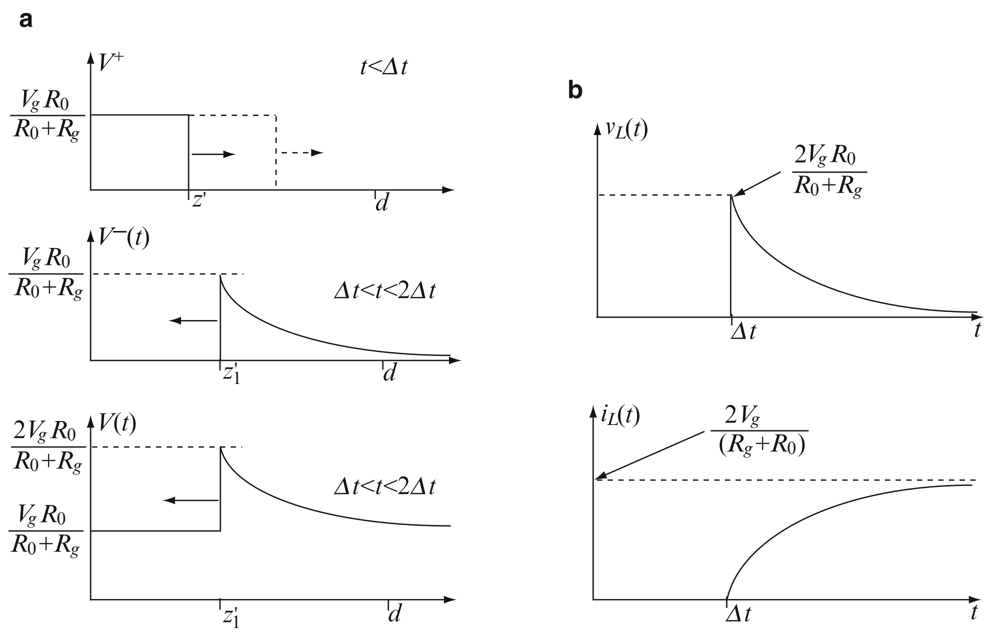


Figure 16.30 (**a**) Forward, reflected, and total voltage on the line in **Figure 16.28** after the capacitance was replaced with an inductance L. (**b**) Load voltage and current on the line for the conditions in (**a**)

Example 16.7 A transmission line with matched generator is 120 m long and terminated by a capacitor as shown in **Figure 16.31**. The characteristic impedance of the line is $Z_0 = 50\ \Omega$, the load capacitance is $C = 100$ pF, and the speed of propagation on the line is c [m/s]. Calculate the voltage at the load for all times.

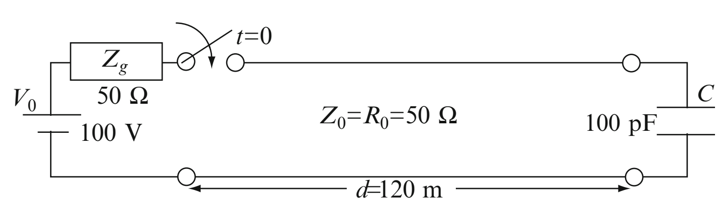


Figure 16.31 A capacitively loaded transmission line with matched generator

Solution: Because the generator is matched ($Z_g = Z_0$), the amplitude of the forward-propagating wave is $V_0/2$. There will be one reflection at the load, and after the backward-propagating wave reaches the generator, there will be no more reflections. Beyond that, the capacitor continues to charge until it reaches steady state. At steady state, the capacitor's voltage equals V_0.

At $t = 0$, the switch is closed and the forward-propagating wave is generated. This travels toward the load at the speed of propagation $v_p = 3 \times 10^8$ m/s. The forward-propagating wave reaches the load at time $\Delta t = d/v_p = 0.4$ μs. During this time, the voltage at the load is zero. The voltage on the line varies from point to point, depending on time, as shown in **Figure 16.32a**. At time $t = \Delta t = 0.4$ μs, the backward-propagating wave is generated. The backward-propagating wave is

$$V^-(t) = V^+\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) = 50\left(1 - 2e^{-\left(t-4\times10^{-7}\right)/5\times10^{-9}}\right) \quad [\text{V}], \quad t \geq \Delta t$$

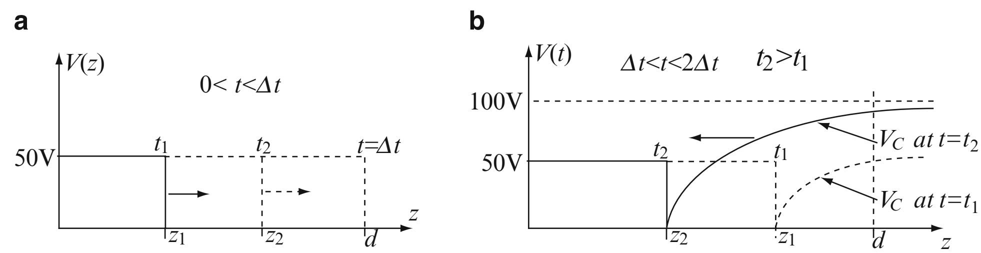


Figure 16.32 (**a**) Propagation of the voltage wave in **Figure 16.31** for $t < \Delta t$. (**b**) The reflected and forward waves for $\Delta t < t < 2\Delta t$

The voltage on the line and load is the sum of the forward- and backward-propagating waves:

$$v(t) = V^+ + V^+\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) = 100\left(1 - e^{-\left(t-4\times10^{-7}\right)/5\times10^{-9}}\right) \quad [\text{V}], \quad t \geq \Delta t$$

This is shown in **Figure 16.32b** for two times, t_1 and t_2, before the backward-propagating wave reaches the generator. The direction of propagation of the waveform is also shown. The capacitor's voltage increases with time until, after considerable time (relative to the time constant) has expired, the capacitor is at a voltage equal to V_g.

After time $t = 2\Delta t = 0.8$ μs, the backward-propagating wave has reached the generator, and since there is no reflection at the generator, the line voltage continues its climb toward steady state as shown in **Figure 16.33a**. The voltage at the load as it varies with time is shown in **Figure 16.33b**. The load voltage is zero between $0 = t < 0.4$ μs. After that it is the sum of the incident and reflected waves and shows steady charging from $v_L = 0$ toward $v_L = V_g$.

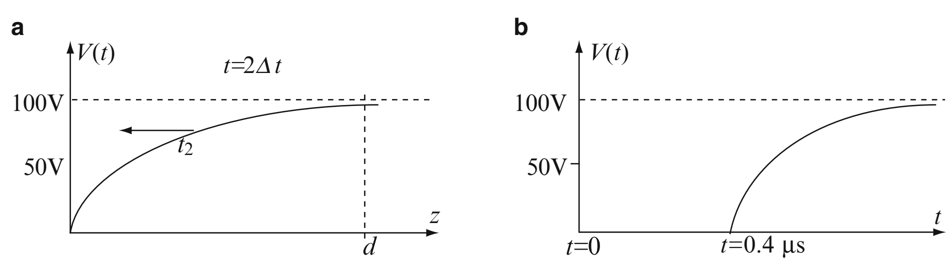


Figure 16.33 (**a**) The total wave at $t = 2\Delta t$ (in **Figure 16.31**). (**b**) Load voltage as a function of time

Exercise 16.4 The line in **Figure 16.31** is given. Find the load current for all times.

Answer

$$i_L(t) = 0, \quad 0 \leq t \leq 0.4 \text{ μs},$$

$$i_L(t) = 2e^{-\left(t-4\times10^{-7}\right)/5\times10^{-9}} \quad [\text{A}], \quad t \geq 0.4 \text{ μs}.$$

Exercise 16.5 The line in **Figure 16.31** is given, but the load is an inductor $L = 100$ μH. Find the load voltage and current for all times.

Answer

$$v_L(t) = 0, \quad 0 \le t \le 0.4\Delta \text{s}, \quad v_L(t) = 100e^{-(t-4\times10^{-7})5\times10^5} \quad [\text{V}], \quad t \ge 0.4\Delta s$$
$$i_L(t) = 0, \quad 0 \le t \le 0.4\Delta s, \quad i_L(t) = 2\left(1 - e^{-(t-4\times10^{-7})5\times10^5}\right) \quad [\text{A}], \quad t \ge 0.4\Delta s.$$

16.8 Initial Conditions on Transmission Lines

There is one additional condition that may exist on a line that we have not considered yet. Until now, we assumed that the line was completely neutral before the transient on the line was introduced. This means, for example, that no current or voltage was present anywhere on the line. In the case of the capacitive or inductive loading, this meant that the solution started with the capacitor or inductor discharged. There are, however, a number of situations in which the conditions are different. For example, a transmission line may operate in its steady-state mode when at some time $t = t_0$, a disturbance occurs. A short on a power line is of this type. Another example may be a line, operating at a given steady-state condition, on which the load is changed suddenly. A line which is at some initial voltage and current at the time the disturbance occurs is called an initially charged line. Treatment of transients on this type of line is performed by superposition of the steady-state line conditions and the conditions due to the transient.

Consider an open line on which the voltage is constant and equals V_0, as shown in **Figure 16.34a**. Now, a load is connected across the terminals at some time $t = 0$. The initial conditions on the initially charged line **(Figure 16.34b)** are

$$V = V_0 \quad [\text{V}], \quad I = 0 \quad [\text{A}] \tag{16.61}$$

When the load resistance is connected at time $t = 0$, the reflection coefficient changes at the load. Initially, the reflection coefficient was 1, but now it changes to a smaller value $\Gamma_L = (R_L - R_0)/R_L + R_0)$ and may be positive or negative. Regardless of the magnitude of the reflection coefficient, a backward-propagating wave is generated, which we denote as V_1^-, indicating that this is the first reflection. The total voltage across the load is the sum of the previously existing condition and the reflected voltage:

$$V_t = V_0 + V_1^- \quad [\text{V}] \tag{16.62}$$

The initial current in the line was zero. Now, however, there must be a current I_1^- reflected from the load. Similarly, from the fact that the current in the line must be continuous, we can write

$$I_1^- = -I_L \quad [\text{A}] \tag{16.63}$$

From the equivalent circuit in **Figure 16.34b**, we have

$$I_L = \frac{V_t}{R_L} = \frac{V_0 + V_1^-}{R_L} = -I_1^- \quad [\text{A}] \tag{16.64}$$

On the other hand, on the line itself, we must have

$$I_1^- = \frac{V_1^-}{Z_0} \quad [\text{A}] \tag{16.65}$$

Thus, we can write

$$V_0 + V_1^- = -I_1^- R_L = -\frac{V_1^- R_L}{Z_0} \quad [\text{V}] \tag{16.66}$$

From this, we obtain the reflected voltage wave as

$$\boxed{V_1^- = -V_0 \frac{Z_0}{Z_0 + R_L} \quad [\text{V}]} \tag{16.67}$$

and from **Eq. (16.20)**, the reflected current wave is

$$\boxed{I_1^- = -\frac{V_0}{Z_0 + R_L} \quad [\text{A}]} \tag{16.68}$$

Now, we can replace the problem by the equivalent circuit at the load as given in **Figure 16.34c**. This equivalent source produces the initial condition for the transient. In other words, this equivalent circuit only exists for the purpose of generating the backward-propagating wave which, in this case, may be viewed as a generator output. Now, we may use the reflection diagram as for any other transient, except that the generator now is at the load (the load generates the input signal that causes the transient). To this, we must add the initial conditions on the line. These points are further clarified in **Example 16.8**.

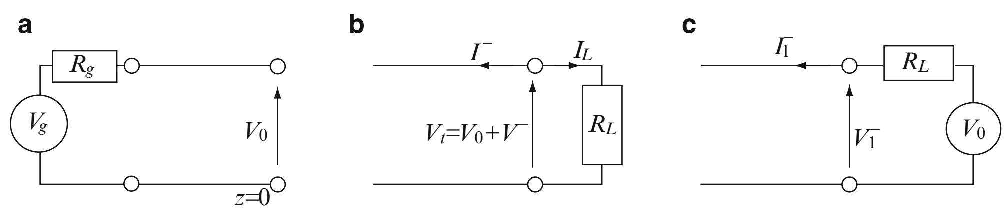


Figure 16.34 (**a**) Open line in steady state. (**b**) A load connected across the line in (**a**). (**c**) Equivalent circuit at the load representing the conditions in (**a**) and (**b**)

Example 16.8 A high-voltage DC (HVDC) line operates at steady state. The voltage on the line is 10^6 V, and the current is zero (no load). The characteristic line impedance is 200 Ω and the generator impedance is 300 Ω. The line length (distance between generator and load) is 1,000 km. Assume a lossless line and the speed of propagation is 2.5×10^8 m/s. A 30 Ω load is connected on the line at $t = 0$:

(a) Calculate the voltage and current at the load at $t = 10$ ms.
(b) Calculate the new steady-state voltage and current on the line.

Solution: Because the load is connected when the line is at the steady-state voltage, the reflection caused by the connection of the load becomes the generator for the transient. This transient is then superimposed on the initial line voltage (or current). The line after connecting the load is shown in **Figure 16.35a** and the equivalent circuit for the transient shown in **Figure 16.35b**:

(a) First, we calculate the transient voltage and current using the circuit in **Figure 16.35b**. The reflection coefficients at the load and generator are

$$\Gamma_L = \frac{R_L - Z_0}{Z_0 + R_L} = \frac{30 - 20}{30 + 20} = 0.2, \quad \Gamma_g = \frac{R_g - Z_0}{Z_0 + R_g} = \frac{30 - 20}{30 + 20} = 0.2$$

The reflected voltage and current due to connection of the load are given in **Eqs. (16.67)** and **(16.68)**:

$$V_1^- = -V_0 \frac{Z_0}{Z_0 + R_L} = -10^6 \times \frac{200}{500} = -0.4 \times 10^6 \quad [\text{V}],$$

$$I_1^- = -\frac{V_0}{Z_0 + R_L} = -\frac{10^6}{500} = -2,000 \quad [\text{A}]$$

The time it takes the current or voltage wave to propagate the length of the line is

$$\Delta t = \frac{L}{v_p} = \frac{10^6}{2.5\times10^8} = 0.004 \quad [\mathrm{s}]$$

These now form the basis of two bounce diagrams shown in **Figures 16.36a** and **16.36b**. Note that the reflection coefficients for the voltage diagram are both positive, whereas for current, we use the negatives of the reflection coefficients as indicated in **Eq. (16.3)** and, more directly, in **Eq. (16.6)**. The propagation starts from the load.

At 10 ms, the waves have bounced once from the generator and once from the load. The transient voltage at the load is

$$V_L(10\,\mathrm{ms}) = V_1 + V_1\Gamma_g + V_1\Gamma_g\Gamma_L = -0.4\times10^6(1+0.2+0.04) = -0.496\times10^6 \quad [\mathrm{V}]$$

To this is added the initial condition on the line of 10^6 V to give the actual load voltage as $10^6 - 0.496\times10^6 = 0.504\times10^6$ V. In other words, the load voltage has dropped to almost half its initial value. The current in the line is

$$I_{line}(10\text{ ms}) = I_1 - I_1\Gamma_g + I_1\Gamma_g\Gamma_L = -2{,}000(1-0.2+0.04) = -1{,}680 \quad [\mathrm{A}]$$

Since the initial current on the line is zero, the total line current at the load also equals −1680 A. The current in the load is in the opposite direction to the line current, as can be seen in **Figure 16.34**. Thus, the load current is 1680 A.

(b) In the steady state, we can use **Eqs. (16.31)** and **(16.32)**. The steady-state voltage and currents on the line due to the transient only are

$$V_\infty = V^+\frac{1+\Gamma_L}{1-\Gamma_L\Gamma_g} \quad\rightarrow\quad V_\infty = V_1^-\frac{1+\Gamma_g}{1-\Gamma_L\Gamma_g} = -0.4\times10^6\frac{1+0.2}{1-0.04} = -0.5\times10^6 \quad [\mathrm{V}]$$

$$I_\infty = I^+\frac{1-\Gamma_L}{1-\Gamma_L\Gamma_g} \quad\rightarrow\quad I_\infty = I_1^-\frac{1-\Gamma_g}{1-\Gamma_L\Gamma_g} = -2{,}000\frac{1-0.2}{1-0.04} = -1{,}666.67 \quad [\mathrm{A}]$$

As previously, we must add to these the initial values at the load. With these and recalling that the current in the load is opposite the current in the line, we get the steady-state voltage and current of the load as

$$V_L = 0.5\times10^6 \quad [\mathrm{V}], \quad I_L = 1{,}666.67 \quad [\mathrm{A}].$$

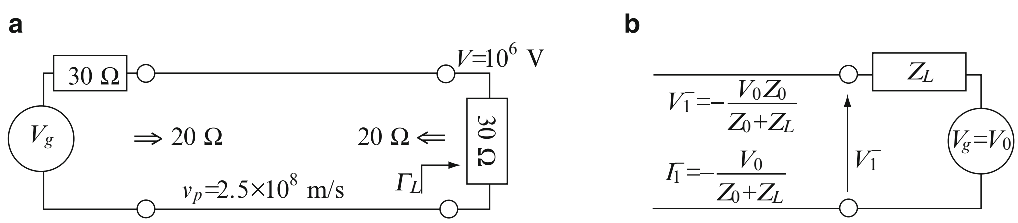


Figure 16.35 (**a**) A load connected across a high-voltage line at steady state. (**b**) The equivalent circuit used to find the transient

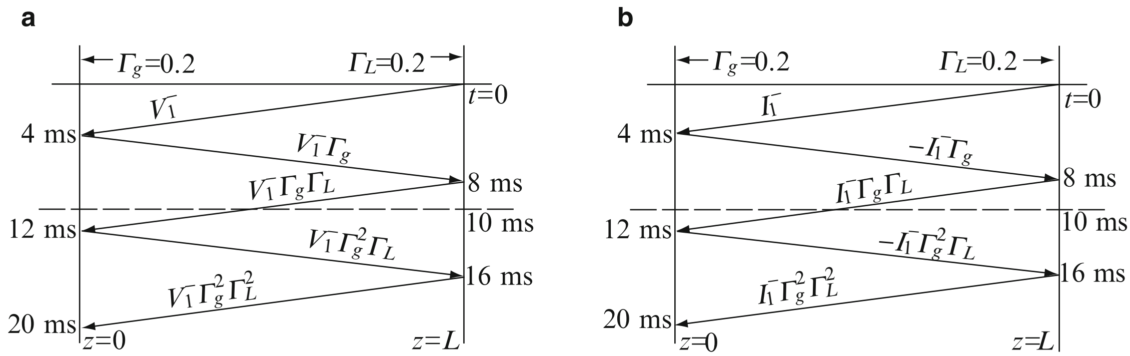


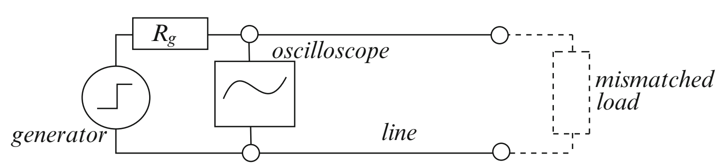
Figure 16.36 (**a**) Voltage reflection diagram for the transient due to **Figure 16.35b**. (**b**) Current reflection diagram for the transient due to **Figure 16.35b**

16.9 Experiments

Experiment 1 (Time Domain Reflectometry. Demonstrates: Reflection and Reflected Waves) Time domain reflectometry can be demonstrated quite easily with an oscilloscope and a signal generator. The best method is to use a step signal generator, but single, narrow pulses or low-duty-cycle pulses may also be used. If a line is available, it can be used. If not, a cable on a spool can be used or you may use the simulated line in **Figure 16.38**. The reflection from the open circuit at the other end of the cable should be recorded on the screen as a pulse following the application of the narrow pulse. Multiple reflections are easily obtained, especially with shorted and open lines. The experiment is shown in **Figure 16.37**.

Time domain reflectometry can be used to detect breaks on lines as well as to detect legal and illegal connections to lines, especially if these are not properly matched. In cable TV maintenance, it is routinely used to locate flaws and bad connections and to detect illegal connections. Time domain reflectometry of transmission lines is an important diagnostics tool because it can also analyze the conditions of the flaw in addition to its location on the line. The type of flaw or discontinuity, impedance on the line, as well as reflection coefficients, standing wave ratios, and the like may be deduced (see **Problems 16.1** and **16.17** through **16.20**).

Figure 16.37 Demonstration of time domain reflectometry



Experiment 2 (Simulated Transmission Line. Demonstrates: Line Properties, Simulated Transmission Line) For most lines, to be able to see the effects discussed in this chapter, the line must be long or the frequency must be high, neither of which is convenient. It is possible to build an artificial or simulated transmission line from simple circuit elements. The series resistance, series inductance, parallel capacitance, and parallel conductance are simulated by resistors, inductors, capacitors, and parallel resistors, respectively. Each group of elements is considered a "cell" or "element" of the line and we may, arbitrarily, associate it with a given length of line such as 1 m or 1 km. Because the components may be chosen individually, any kind of line may be easily simulated. **Figure 16.38** shows a simulated transmission line that may be used for a number of experiments. The line parameters may be changed by simply changing the components in each cell, and different lines may be connected to simulate discontinuities. Lossless lines are simulated by using low-resistance inductors and low-loss capacitors.

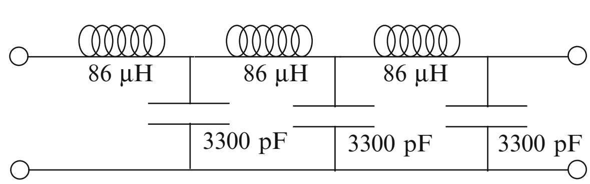


Figure 16.38 A simulated transmission line

16.10 Summary

Following the frequency domain analysis in **Chapters 14** and **15**, this chapter discusses transient analysis and propagation of pulses on transmission lines. Again, the dominant issues are the reflection and transmission coefficients at discontinuities on transmission lines but the analysis is in the time domain.

Narrow Pulses Narrow pulses propagate, attenuate as they propagate, reflect, and transmit at all discontinuities. The forward-propagating waves generated by the generator (such as when closing a switch):

$$V^+ = V_g \frac{Z_0}{Z_0 + Z_g} \quad [\text{V}] \qquad I^+ = \frac{V_g}{Z_0 + Z_g} \quad [\text{A}] \tag{16.2}$$

When the pulse reaches the load **(Figure 16.3)**, the first reflection is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \quad (16.3) \qquad V_1^- = V^+\Gamma_L \quad [\text{V}] \quad (16.5) \qquad I_1^- = -\frac{V^+\Gamma_L}{Z_0} \quad [\text{A}] \tag{16.6}$$

The total voltage and current at load during the length of the pulse after first reflection:

$$V_{L1} = V^+(1 + \Gamma_L) \quad [\text{V}] \quad (16.7) \qquad I_{L1} = \frac{V^+}{Z_0}(1 - \Gamma_L) \quad [\text{A}] \tag{16.8}$$

Back at the generator, the first reflection of the backward-propagating wave:

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} \quad (16.9) \qquad V_1^+ = \Gamma_L\Gamma_g V^+ \quad [\text{V}], \quad I_1^+ = \frac{V^+\Gamma_L\Gamma_g}{Z_0} \quad [\text{A}] \tag{16.10}$$

Notes:

(1) Reflections repeat indefinitely unless the load and/or generator are matched.
(2) The process stops at a matched location (no reflection).
(3) Total voltage or current at a given location during the width of the pulses is the sum of the voltages (currents) at that point (load and generator in particular).
(4) Attenuation (if any) is cumulative—it only depends on the total distance traveled by the pulse.

Step Pulses The step pulse propagates, reflects, and transmits at any discontinuity on the line.

Reflection Diagram A space–time diagram showing the propagation of the wave in space and time:

(1) Time is horizontal, space is vertical (see **Figures 16.13** and **16.14**).
(2) Voltages and currents reflected from all discontinuities are traced through time and space.
(3) The voltage (or current) at any point on the line is the sum of all voltages (or currents) at that location up to that time.

Steady-State Voltages and Currents on Lossless Lines

$$V_\infty = V^+\frac{1 + \Gamma_L}{1 - \Gamma_L\Gamma_g} = V_g\frac{Z_L}{Z_g + Z_L} \quad [\text{V}] \tag{16.30, 16.33}$$

$$I_\infty = I^+\frac{1 - \Gamma_L}{1 - \Gamma_L\Gamma_g} = \frac{V_g}{Z_g + Z_L} \quad [\text{A}] \tag{16.31, 16.34}$$

Finite-length pulses

(1) Finite-length pulses are viewed as superposition of positive and negative step pulses (**Figure 16.19**).
(2) Treat the positive going step pulse and the negative going step pulse separately using the reflection diagram and add the results together (see **Example 16.5**).
(3) Can also generate shaped pulses by superposition of pulses of various amplitudes and widths (**Example 16.5**).

Reactive Loads The reflection coefficient is not properly defined—it depends on amplitude.

Calculate the reflected voltage by solving a differential equation at the reflecting point (for example, at the load) as follows.

For Capacitive Loading

$$i_L(t) = C\frac{d}{dt}(v_L(t)) \quad [\text{A}] \tag{16.47}$$

Given a transmission line with characteristic impedance R_0, internal generator impedance R_g, and a capacitor C as load, the reflected voltages and currents at the load are [see **Eq. (16.46)** for calculation of V^+]

$$V_1^-(t) = V^+\left(1 - 2e^{-(t-\Delta t)/R_0C}\right) = \frac{V_gR_0}{R_0+R_g}\left(1-2e^{-(t-\Delta t)/R_0C}\right) \quad [\text{V}] \tag{16.53}$$

$$I_1^-(t) = \frac{-V^-(t)}{R_0}\left(1-2e^{-(t-\Delta t)/R_0C}\right) = -\frac{V_gR_0}{R_0(R_0+R_g)}\left(1-2e^{-(t-\Delta t)/R_0C}\right) \quad [\text{A}] \tag{16.54}$$

For Inductive Loading

$$v_L(t) = L\frac{d}{dt}(i_L(t)) \quad [\text{V}] \tag{16.55}$$

Given a forward-propagating voltage V^+, the reflected voltage and current at the load are

$$V_1^-(t) = V^+\left(2e^{-(t-\Delta t)R_0/L} - 1\right) = \frac{V_gR_0}{R_0+R_g}\left(2e^{-(t-\Delta t)R_0/L}-1\right) \quad [\text{V}] \tag{16.59}$$

$$I_1^-(t) = -\frac{V_gR_0}{R_0(R_0+R_g)}\left(2e^{-(t-\Delta t)R_0/L}-1\right) \quad [\text{A}] \tag{16.60}$$

These then propagate on the line and may reflect again off the generator (unless it is matched).

Initial Conditions on Lines A line at steady state is characterized by a constant voltage V_0 and current I_0. Change in loading then adds reflected voltages and currents which take the line to a new steady state after these generated voltages and currents settle. The reflected voltage and current due to connection of a load, R_L, to an open line with characteristic impedance Z_0 are

$$V_1^- = -V_0\frac{Z_0}{Z_0+R_L} \quad [\text{V}] \tag{16.67}$$

$$I_1^- = -\frac{V_0}{Z_0+R_L} \quad [\text{A}] \tag{16.68}$$

These now propagate on the line exactly as any step voltage and current and add to the existing conditions on the line. Any discontinuity will create additional reflections until a new steady state is achieved.

Time Domain Reflectometry In this method, often used for testing of line conditions, a pulse is sent on the line and the reflected pulse is received after a time Δt. The distance to the discontinuity that caused the reflection is $d = v\Delta t/2$ where v is the speed of propagation on the line. By measuring time one can identify the location of discontinuity provided the speed of propagation on the line is known.

Problems

Propagation of Narrow Pulses on Finite, Lossless, and Lossy Transmission Lines

16.1 **Narrow Pulses on Mismatched Line**. A generator is matched to a line. A single, narrow pulse is applied to the line. The load equals $2Z_0$ [Ω], where $Z_0 = 50$ Ω is the characteristic impedance of the line. If the pulse is 20 ns wide and the delay on the line (time of propagation to load) is 100 ns, calculate the line voltage and current at the load for $t > 0$ for a generator voltage of 1 V.

16.2 **Narrow Pulses on Mismatched Line**. A generator with an internal impedance $2Z_0$ [Ω] is connected to a line of characteristic impedance $Z_0 = 50$ Ω. A single, narrow pulse is applied to the line. The load equals $2Z_0$ [Ω]. If the pulse is 20 ns wide and the delay on the line is 100 ns, calculate the line voltage and current for $t > 0$ for a generator voltage of 1 V.

16.3 **Application: Transients in Digital Circuits.** Two sensors are connected as inputs to an AND gate as shown in **Figure 16.39**. The lines have characteristic impedance of 50 Ω. Input impedance to each input of the gate is 50 Ω. The sensors supply an open circuit voltage of 10 V and the AND gate has a threshold of 3.25 V (i.e., if both inputs are above this value, the output is 5 V; if one or both are below 3.25 V, the output is zero). One line is 10 m long, the second is 100 m long, and the speed of propagation is $0.1c$ [m/s]. Each of the sensors sends a single pulse, 50 ns wide at $t = 0$. The sensors are matched to the line:

(a) Calculate the gate output for $t > 0$.
(b) What must be the minimum pulse width for the output to ever be "1"? What are your conclusions from this result?

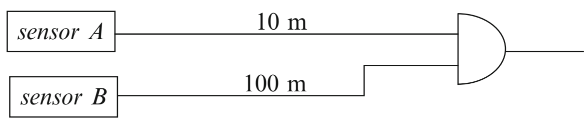


Figure 16.39

16.4 **Application: Reflectometry (Narrow Pulses)**. A lossless cable TV coaxial transmission line is matched to both generator and load. As a routine test, a signal is applied to the input and sent down the line. The distance to the receiver is known to be $d = 1$ km. The speed of propagation on the line is $v_p = c$ [m/s], and the characteristic impedance on the line is $Z_0 = 75$ Ω:

(a) The signal in **Figure 16.40a** is obtained on the oscilloscope screen. If $\Delta t = 0.1$ μs, what happened to the line and at what location?
(b) The signal in **Figure 16.40b** is obtained on the oscilloscope screen. If $\Delta t = 0.2$ μs, what happened on the line and at what location?

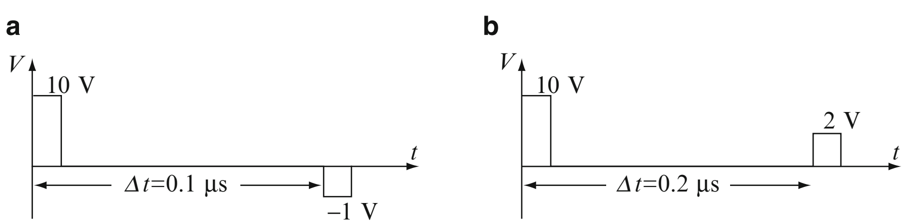


Figure 16.40

16.5 **Application: Reflections on Lossy Line.** The cable in **Problem 16.4** is given again. However, now the line is considered distortionless, with an attenuation constant of 0.001 Np/m:

(a) The signal in **Figure 16.40a** is obtained on the oscilloscope screen. If $\Delta t = 0.1$ μs, what happened to the line and at what location?
(b) The signal in **Figure 16.40b** is obtained on the oscilloscope screen. If $\Delta t = 0.2$ μs, what happened on the line and at what location?
(c) Compare the location of the fault on the line and magnitude of fault impedance with those for the lossless line in **Problem 16.4**.

Transients on Transmission Lines: Long Pulses

16.6 Transients on an Open Line. A lossless open transmission line is given as shown in **Figure 16.41**. The line is 10 m long and has a capacitance of 200 pF/m and inductance of 0.5 μH/m. Calculate the transient voltage at a distance of 5 m from the DC source:

(a) 0.5 μs after closing the switch.
(b) 50 μs after closing the switch.

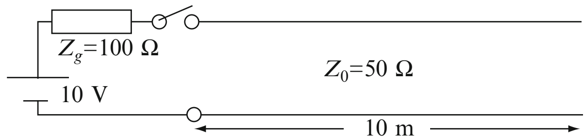


Figure 16.41

16.7 Line Voltage on Long, Loaded Line. A line is very long and the speed of propagation on the line is 10^8 m/s. Assume the ideal DC source has been switched on. The voltage wave reaches the load at time t_0. Calculate the voltage at point $A - A'$ (2 m from the load) for $t > t_0$ and for times $t < t_0$ (**Figure 16.42**).

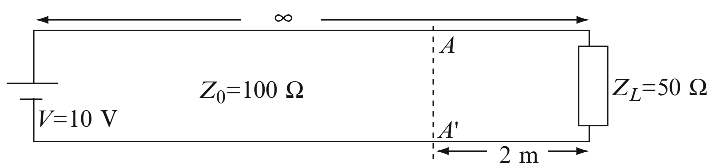


Figure 16.42

16.8 Transient and Steady-State Voltages on Lossless Line. A lossless transmission line of length d is given as in **Figure 16.43**. The transmission line has a capacitance per unit length of C_0 [F/m] and an inductance per unit length of L_0 [H/m]. The switch is closed at time $t = 0$. Given: $L_0 = 10$ μH/m, $C_0 = 1{,}000$ pF/m, $d = 1{,}000$ m, $R_g = 100$ Ω, $R_L = 50$ Ω, and $V_0 = 100$ V:

(a) Calculate the steady-state voltage on the line.
(b) Calculate the steady-state current in the line.
(c) How long does it take the voltage to reach steady state at the load?
(d) How long does it take the voltage to reach steady state at the generator?

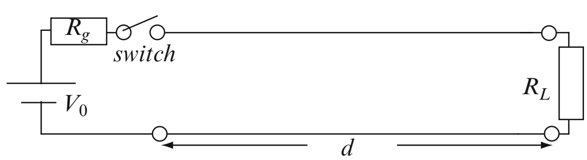


Figure 16.43

Transients on Transmission Lines: Finite-Length Pulses

16.9 Transient Due to a Single Square Pulse. The transmission line in **Figure 16.44** is given. The generator supplies a single pulse as shown. Calculate:

(a) The voltage and current at the generator 10 μs after the pulse began.
(b) The steady-state current and voltage on the line.

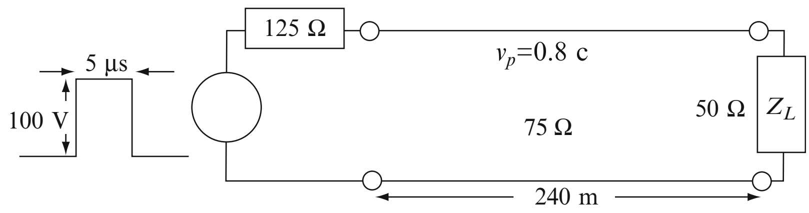


Figure 16.44

16.10 Transient Due to a Single Square Pulse on Lossy Line. The circuit in **Figure 16.44** is given. In addition, the line has an attenuation constant $\alpha = 0.0001$ Np/m. Assume the line is distortionless and calculate the voltage and current at the generator 10.5 μs after the pulse began.

Reflections from Discontinuities

16.11 Reflections from Discontinuities. Three sections of lines are connected as shown in **Figure 16.45**. The propagation time on each section is indicated:

(a) If the load R_L is matched, but the generator's impedance is 50 Ω, calculate the line voltage at g, L, and on both sides of the discontinuities a and b, 45 ns after the switch is closed.

(b) Same as **(a)** but if both the source and load are matched.

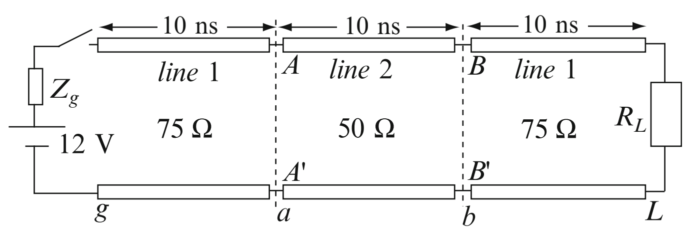


Figure 16.45

16.12 Reflections from Discontinuities. Use the same figure and data as in **Problem 16.11**. The load now is a short circuit. Given a matched source, calculate the voltage and current at g, L, and on both sides of the discontinuities a and b, 45 ns after the switch is closed.

Reactive Loading

16.13 Application: Capacitively Loaded Transmission Line. A long lossless transmission line with a characteristic impedance of 50 Ω is terminated with a 1 μF capacitor. The length of the line is 100 m and the speed of propagation on the line is $c/3$ [m/s]. At $t = 0$, a 100 V matched generator is switched on. Calculate and plot:

(a) The load voltage and current for $t > 0$.

(b) The line voltage and current at any point on the line for $t > 0$.

16.14 Application: Inductively Loaded Transmission Line. A long lossless transmission line with characteristic impedance of 50 Ω is terminated with a 1 μH inductor. The line is 10 km long and the speed of propagation on the line is $c/3$ [m/s]. At $t = 0$, a 100 V matched generator is switched on:

(a) Calculate and plot the load voltage and current for $t > 0$.

(b) Calculate and plot the line voltage and current at any point on the line for $t > 0$.

16.15 Application: Initially Charged Line. A 300 m long, lossless transmission line has characteristic impedance of 75 Ω and speed of propagation of $c/3$ [m/s]. The transmission line is matched at the generator and is open ended. The generator's voltage is 100 V. After the line has reached steady state, the generator is disconnected and a resistor $R = 125$ Ω is connected across the open end. Calculate and plot the voltage on and the current in R.

16.16 Application: Initially Charged Line. A 100 m long lossless transmission line has characteristic impedance of 75 Ω and speed of propagation of $0.2c$ [m/s]. The transmission line is matched at the generator and is open ended. The generator's voltage is 100 V. After the line has reached steady state, the generator is disconnected and a resistor $R = 125$ Ω is connected across the open end:

(a) Calculate the voltage and current in R.
(b) How long does it take for the voltage on R to be below 1 V?

Time Domain Reflectometry

16.17 Application: Time Domain Reflectometry. An underground cable used for transmission of power has developed a fault. The speed of propagation on the line is known and equal to v_p [m/s]. To locate the fault before starting to dig, time domain reflectometry is performed. A 1 V step pulse is applied to the input with matched impedance and the output in **Figure** 16.46a is obtained on the oscilloscope. The characteristic impedance of the cable is $Z_0 = 50$ Ω. Use $v_p = 0.2c$ [m/s] and calculate:

(a) The location of the fault.
(b) Type of fault: calculate the impedance on the line at the fault.

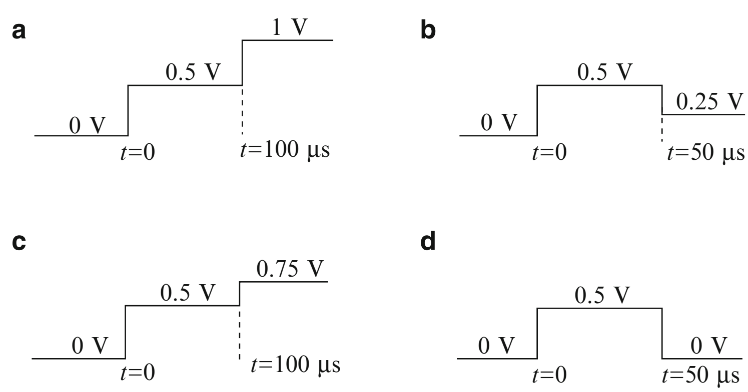


Figure 16.46

16.18 Application: Time Domain Reflectometry. The measurement in **Problem** 16.17 is performed on a line and the signal in **Figure** 16.46b is recorded on the time domain reflectometer. Using the data in **Problem** 16.17, calculate:

(a) The location of the fault.
(b) Type of fault: calculate the impedance on the line at the fault.

16.19 Application: Time Domain Reflectometry. The measurement in **Problem** 16.17 is performed on a line and the signal in **Figure** 16.46c is recorded on the time domain reflectometer. Using the data in **Problem** 16.17, calculate:

(a) The location of the fault.
(b) Type of fault: find the impedance on the line at the fault.

16.20 Application: Time Domain Reflectometry. The measurement in **Problem** 16.17 is performed on a line and the signal in **Figure** 16.46d is recorded on the time domain reflectometer. Using the data in **Problem** 16.17, calculate:

(a) The location of the fault.
(b) Type of fault: find the impedance on the line at the fault.