Electromagnetic Waves and Propagation

12

As to the properties of electromagnetic radiation, I need first of all to come up with a little more than just words to the idea that a changing magnetic field makes an electric field, a changing electric field makes a magnetic field and that this pumping cycle produces an electromagnetic wave…

J. Robert Oppenheimer (1904–1967)
physicist in *The Flying Trapeze*, Oxford University Press, 1964

12.1 Introduction

After summarizing Maxwell's equations in **Chapter 11**, we are now ready to discuss their implications. In particular, we will deal directly or indirectly with the displacement current term in Maxwell's equations.

We have alluded to the fact that displacement currents are responsible for the wave or propagating nature in the field equations. Although we have some understanding of what a wave is, this understanding probably does not extend to electromagnetic waves. This will be our first task: to understand what electromagnetic waves are, why they must exist, and, later, to define the properties of the waves. The applications resulting from this new view of electromagnetics are vast and exciting, and we will have a chance to discuss some of them here and in the remainder of this book.

12.2 The Wave

What is a wave? Why is it important? What does it add to the physics of electromagnetics that was not present in the time-dependent field as discussed in **Chapter 10**? We will try to answer the first two questions directly whereas the third will become self-evident as a result.

Consider, first, an example: An earthquake, centered 100 km from a city, causes some damage in the city. For this damage to occur, there must be a mechanism by which energy generated at the center of the earthquake is propagated. The earthquake produces stresses in the Earth's crust and these stresses are relieved by propagating the energy and dissipating it over large areas of the Earth's surface, causing Earth movement at relatively large distances from the source. The same effect is felt when pounding with a hammer on a piece of wood or any other material. The material flexes under the pressure of the hammer and the effect can be felt at a distance from the source.

A few observations on the above scenario are useful here:

(1) As the distance from the center increases, the magnitude of Earth movement is reduced (attenuated).
(2) Recording of the earthquake on a seismograph looks as in **Figure 12.1a**, showing a repetitive movement of the crust, at some frequency or range of frequencies.
(3) The tremor is felt at different locations at different times; propagation of the earthquake is at finite (relatively low) speeds.

N. Ida, *Engineering Electromagnetics*, DOI 10.1007/978-3-319-07806-9_12,

The earthquake is an elastic wave because it requires that the intervening material sustains stress. In other words, if a location within the range of the earthquake could be isolated from the crust with a vacuum gap, air, or any other flexible material (such as rubber), the earthquake would not affect that location. For example, you would not be able to feel an earthquake while flying over an affected area. This is because air can sustain very little stress. We refer to this effect as attenuation: The elastic wave is attenuated rapidly in air. On the other hand, earthquakes propagate very well in water.

Another example of a wave is the motion of a string, fixed at one end to a post and the other end free to move. The free end moves up and down at a constant rate as shown in **Figure 12.1b**. The amplitude at the free end depends on how far the end moves, and the frequency of the wave depends on the rate of motion. In the process, the string oscillates, and energy is transferred through the motion. For example, if you were to hold your hand in the path of the string, you would feel the motion as the string hits your hand. The amplitude becomes smaller closer to the fixed end. At the fixed end, the amplitude is constrained to zero.

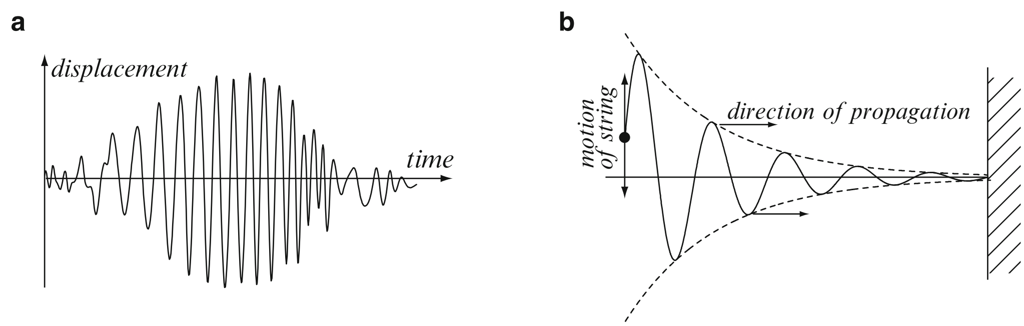

Figure 12.1 Two types of waves. (**a**) Seismograph recording of the wave nature of an earthquake. (**b**) Wave motion generated by moving the free end of a string up and down

Now, we can define the wave and its properties based on the above observations:

A wave is a disturbance in the surrounding medium with the following properties:

(1) The disturbance occurs in space and must be time dependent. Hitting a material with a hammer produces a wave; the action is a disturbance in space and is time dependent. Other examples are the operation of a loudspeaker, turning on the light, moving a paddle in water, plucking a string, etc.

(2) The disturbance can be a single event (hitting a nail with a hammer), repetitive (a paddle moving in water to generate waves), or time harmonic (for example, sinusoidal motion of the loudspeaker diaphragm).

A wave propagates in the medium with the following properties:

(1) The disturbance, which we will now call a wave, propagates in the medium or across media, at finite speeds. A sound wave propagates in water at a speed of about 1,500 m/s, whereas in air, it propagates at about 340 m/s. Light propagates in air at 3×10^8 m/s. Earthquakes propagate at speeds between about 2,000 and over 8,000 m/s, depending on the composition of the Earth's crust and location in the crust. In seawater, the propagation is at about 1,500 m/s, again varying with depth. Tsunamis (tidal waves), generated by earthquakes, propagate at speeds between about 250 km/h and over 1,000 km/h depending on the depth of the ocean.

(2) Waves propagate with attenuation. After being generated at the source, the wave propagates outward from the source and, in the process, loses its "strength." Sound becomes fainter the further we are from the source, whereas the wake of a boat becomes weaker at larger distances. The reduction in amplitude of a wave may be either because of losses (such as friction or absorption) in the material or simply because the power in the wave is spread in a continually increasing volume as the wave propagates. Attenuation due to losses is very much material dependent, as was the speed of propagation. For example, waves in water and waves in oil are attenuated differently. Sound propagates farther in cold, dense air and to shorter distances in warm air. Light is attenuated more by particulate matter in the air and is attenuated very rapidly by most solids. There are large variations in attenuation in various materials. For this reason, we assume waves propagate in all materials, the differences between them being characterized by the attenuation. For example, light is assumed to propagate in solids but with such high attenuation that a thin layer of the solid seems to block light. However, a thin enough layer would be transparent.

(3) A wave transports energy. We may say that loud noises "hurt" or are "painful." The only way this can happen is if energy is transferred by the wave to our ear drums. Similarly, a very loud sound, such as a sonic boom or an explosion, may shatter windows: energy is coupled from the source through the sound waves to the window.

(4) Propagation of the wave is directional: waves propagate away from a boat and away from a loudspeaker. Electromagnetic waves transmitted from a satellite propagate from satellite to Earth.
(5) Waves can be reflected, transmitted, refracted, and diffracted. Reflection, transmission, refraction, and diffraction of light are well-known examples to these properties of electromagnetic waves.
(6) There are different types of waves with different properties. In the example of earthquakes, the *P* (primary or compressional) wave is generated by alternate compression and stretching of material. The *S* (shear or transverse) wave is generated by the motion of particles perpendicular to the direction of propagation. There are also surface waves, and each travels at different speeds. The same general principles apply to electromagnetic waves, including the existence of surface waves.

To see how wave properties manifest themselves, consider the motion of a tight string. When you pluck the string of a guitar, it moves from side to side, generating a sound wave. There are, in fact, two effects here. One is the motion of the string itself, which is a wave motion. The second is the change in air pressure generated by the string's motion, which are the sound waves we hear. The sound (frequency) depends on the size of the string (both length and thickness) and the amplitude depends on the displacement. The wave produced in the string in **Figure 12.2**, in the form of displacement of the string, y, is described by the following equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{Tg}{w}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{12.1}$$

where T is tension in the string [N], g is the gravity acceleration [m/s^2], and w is weight per unit length of the string [N/m]. The term Tg/w has units of [m^2/s^2] and is, therefore, a velocity squared. This is the velocity of propagation of the wave in the string. This equation is a scalar wave equation and its solution should be familiar from physics. The important point is that it defines the form of a wave equation; the function (displacement in this case) is both time dependent and space dependent. There are other terms that may exist (such as a source term or a loss term), but the two terms above are essential. The field so represented is a wave and has all the properties described above. **Equation (12.1)** is normally written in more convenient forms as

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{12.2}$$

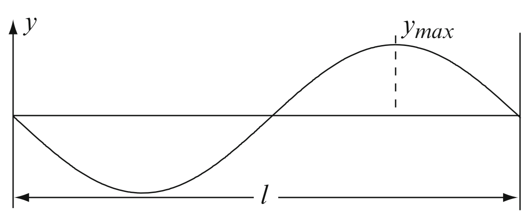

Figure 12.2 Wave motion of a tight string of length l

What about the solution to this equation? This can be obtained in a number of ways. One is by separation of variables. The second is to introduce two new independent variables $\xi = x - vt$ and $\eta = x + vt$, substitute these into the wave equation, and perform the derivatives. Then, by integration on the two new variables, we obtain a general solution of the form[1]

$$y(x,t) = g(x - vt) + f(x + vt) \tag{12.3}$$

where $g(x,t)$ and $f(x,t)$ are arbitrary functions, which describe the shape of the wave. These may be the displacements of the string at any given time and location. For example, $\sin(x - vt)$ and $\cos(x - vt)$ may be appropriate functions. We can get a better feel for what the solution means by taking a very long string (such as a wire between two posts).We will assume here that the string is infinite. Now, we create a disturbance such as plucking the string at a time $t = 0$. This gives the initial condition $y(x,0) = g(x) + f(x)$. Consider, for example, the disturbance shown in **Figure 12.3b**, created by moving the string as shown. If we let go, the disturbance moves in both directions at a velocity v. After a time t_1, the disturbances have moved to the right and left a distance vt_1 as shown. The disturbances propagating in the positive and negative x directions propagate

[1] This solution is known as the D'Alembert's solution of the wave equation.

away from the source and are called forward-propagating waves. For a vector field, such as the electric or magnetic field, an equation equivalent to **Eq. (12.1)** is the vector wave equation:

$$\frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\partial^2 \mathbf{A}}{\partial z^2} \quad \text{or} \quad \frac{\partial^2 \mathbf{A}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{12.4}$$

where **A** stands for any of the field vectors (**E**, **H**, etc.), v is the speed of propagation of the wave, and, in this case, the wave is assumed to propagate in the z direction.

The electromagnetic wave equation will be solved in phasors, in the frequency domain rather than in the time domain. The propagation of the wave is real nonetheless. Speed of propagation, amplitudes, and all other aspects of the wave are similar to the above simple problem, although the displacement of the string will be replaced by the amplitude of the electric or magnetic fields and the propagation will be in space since the field is defined in space. On the other hand, we will talk about propagation in a certain direction in space exactly like the propagation along the string and a material-related velocity.

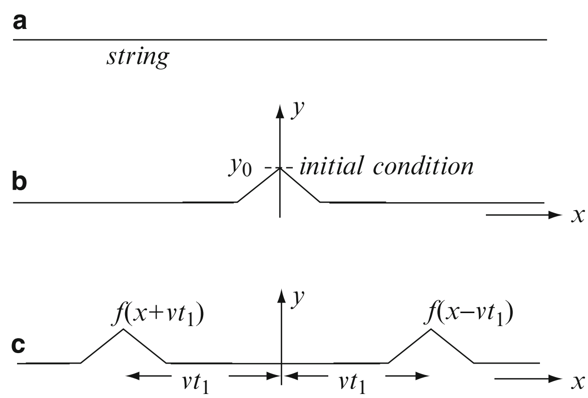

Figure 12.3 Propagation of a disturbance in a tight string. (**a**) String before the disturbance occurs. (**b**) The disturbance is introduced at $x = 0$, $t = 0$. (**c**) The disturbance propagates in the positive and negative x directions at speed v

Example 12.1 Show that the solution $y(x,t) = (1/2)[(x - vt)^2 + (x + vt)^2]$ is a solution to the scalar wave equation in **Eq. (12.1)**.

Solution: Substitution of the given solution in **Eq. (12.1)** and performing the required derivatives should result in an equality. Starting with the wave equation:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

Performing the left-hand time derivatives, we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{2}(x - vt)^2 + \frac{1}{2}(x + vt)^2\right) = 2v^2$$

On the right-hand side, we have

$$v^2 \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}(x - vt)^2 + \frac{1}{2}(x + vt)^2\right) = 2v^2$$

Because the two sides are identical, the solution satisfies the wave equation.

Example 12.2 An electric field intensity is given in free space as $\mathbf{E} = \hat{\mathbf{x}}\,100\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right)$ [V/m] where μ_0 and ε_0 are the permeability and permittivity of free space, respectively:

(a) Calculate the amplitude, frequency, and speed of propagation of the wave.
(b) Show that this solution is of the same form as that in **Eq. (12.3)**.
(c) What is the direction of propagation?

Solution: The given electric field intensity has a single component in the x direction, but the component is independent of x. Therefore, if it is a solution to the wave equation, it must be a solution to an equation of the form of **Eq. (12.4)** because the solution is a vector. To calculate the wave properties, we substitute the electric field intensity in **Eq. (12.4)**.

(a) The relevant wave equation in this case is

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Substituting the electric field intensity and performing the derivatives:

$$\hat{\mathbf{x}}\frac{\partial^2\left[100\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right)\right]}{\partial z^2} - \hat{\mathbf{x}}\frac{1}{v^2}\frac{\partial^2\left[100\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right)\right]}{\partial t^2}$$
$$= -\hat{\mathbf{x}}\,100\times\left(10^6\right)^2\mu_0\varepsilon_0\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right) + \hat{\mathbf{x}}\frac{1}{v^2}100\times\left(10^6\right)^2\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right) = 0$$

After dividing both sides by $100\times\left(10^6\right)^2\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right)$, we get

$$\mu_0\varepsilon_0 = \frac{1}{v^2} \quad\rightarrow\quad v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Thus, the speed of propagation of the wave must be

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c = 3\times 10^8 \quad [\mathrm{m/s}]$$

From the electric field intensity itself, we can write the amplitude as $E_0 = 100$ V/m and frequency as $f = 10^6/2\pi$ [Hz] by simply writing the solution as

$$\mathbf{E} = \hat{\mathbf{x}}\,100\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right) = \hat{\mathbf{x}}\,E_0\cos(\omega t + \varphi) \quad [\mathrm{V/m}].$$

(b) The electric field intensity may be written as follows:

$$\mathbf{E} = \hat{\mathbf{x}}\,100\cos\left(10^6 t - 10^6\sqrt{\mu_0\varepsilon_0}z\right) = \hat{\mathbf{x}}\,100\cos\left[-\omega\sqrt{\mu_0\varepsilon_0}\left(z - \frac{t}{\sqrt{\mu_0\varepsilon_0}}\right)\right] = \hat{\mathbf{x}}\,g(z - vt) \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

This is clearly of the same form as **Eq. (12.3)**, except that now the solution is a vector and only the first part in **Eq. (12.3)** is present (the second part is zero).

(c) The direction of propagation of the wave can be determined from comparison of the electric field intensity (the solution) with the general solution in **Eq. (12.3)**. In this case, the direction of propagation is in the z direction. Note, in particular, that the electric field intensity is directed in the x direction but propagates in the z direction.

Exercise 12.1 (**a**) Show that the function $y(x,t) = \cos(x - wt) + \cos(x + wt)$ is a solution to the scalar wave equation. (**b**) What must be the speed of propagation of the wave, in this case?

Answer (**b**) $v = w$ [m/s].

12.3 The Electromagnetic Wave Equation and Its Solution

Based on the introduction of the displacement currents in Ampere's law, Maxwell predicted the existence of propagating waves, a prediction that was verified experimentally in 1888 by Heinrich Hertz. This prediction was based on the nature of the equations one obtains by using Maxwell's equations. We will show here that Maxwell's equations result, in general, in wave equations. These can be written in a number of useful forms, each useful under certain conditions. The solutions to the electromagnetic wave equations lead to a number of useful definitions, including phase velocity, wave impedance, and others.

Two types of equations will be discussed. One is the source-free wave equation, also called a ***homogeneous wave equation***. The second is a complete equation, including source terms, and is called a ***nonhomogeneous wave equation***. We first use the equations in the time domain, but most of our work here and in the following chapters will be in terms of phasors and the time-harmonic wave equation. It should also be remembered that homogeneity here relates to the form of the equation and should not be confused with material homogeneity, which merely states that material properties are independent of position.

12.3.1 The Time-Dependent Wave Equation

How do we know that Maxwell's equations in fact represent wave equations? If they do, how do we show that is the case? A hint to what needs to be done is the form in **Eq. (12.1)**; we need to rewrite Maxwell's equations in this form. To do so, we must obtain a second-order equation in time and space, in terms of a single variable. In fact, we have already done so in **Chapter 11**. There, we wrote Maxwell's equations in terms of the magnetic vector potential [see **Eq. (11.50)**] as

$$\nabla^2 \mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} = -\mu \mathbf{J} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{12.5}$$

This equation is of the same form as **Eq. (12.1)** except that now the magnetic vector potential varies in all three spatial directions and, in addition, a source term (current density) is included. This is a nonhomogeneous wave equation and is much more general than **Eq. (12.1)**. If the source does not exist, we obtain the source-free or homogeneous wave equation for the magnetic vector potential:

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathbf{A}}{{}^2\partial t^2} \tag{12.6}$$

This particular form of the equations is only one possible form. Other potential functions or the various field variables themselves may be used to obtain similar wave equations. The principle is to substitute the corresponding variable into Maxwell's equations and manipulate the equation until the resulting equation is in terms of a single variable.

Example 12.3 Obtain a wave equation in terms of the electric scalar potential, V.

Solution: We start with Faraday's law (Maxwell's first equation) and use the Lorenz condition given in **Eq. (11.49)**. From Faraday's law, we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \quad \to \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \quad \to \quad \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

The electric field intensity is [see also **Eqs. (11.41)** through **(11.45)**]

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

This relation is substituted in Maxwell's third equation

$$\nabla \cdot \mathbf{D} = \rho_v \quad \to \quad \nabla \cdot \mathbf{D} = \varepsilon \nabla \cdot \mathbf{E} = \varepsilon \nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right) = \rho_v$$

Expanding the terms in parentheses and dividing by ε on both sides of the equation, we obtain

$$-\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \nabla \cdot (\nabla V) = \frac{\rho_v}{\varepsilon}$$

Now, we multiply both sides by -1, interchange between the divergence and time derivative and recall that $\nabla \cdot (\nabla V) = \nabla^2 V$ [vector identity in **Eq. (2.132)**]:

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) + \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

The Lorenz condition in **Eq. (11.49)** [$\nabla \cdot \mathbf{A} = -\mu\varepsilon(\partial V/\partial t)$] is now used to eliminate the magnetic vector potential. Substituting this for $\nabla \cdot \mathbf{A}$ and rearranging terms gives

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\varepsilon \frac{\partial V}{\partial t}\right) = \nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}$$

If $\rho_v = 0$, we obtain the homogeneous wave equation in terms of the electric scalar potential V. The nonhomogeneous and homogeneous wave equations are

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \quad \text{(nonhomogeneous)}$$

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0 \quad \text{(homogeneous)}.$$

Example 12.4 Obtain a source-free wave equation for the magnetic field intensity, **H**.

Solution: To obtain a wave equation in terms of **H**, we start with Ampere's law in **Eq. (11.6)**

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} + \frac{\partial(\varepsilon \mathbf{E})}{\partial t}$$

where we substituted $\mathbf{D} = \varepsilon \mathbf{E}$. Now, we seek to substitute for **E**, in order to eliminate it. To do so, we use Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \mu \mathbf{H}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

where the constitutive relation $\mathbf{B} = \mu\mathbf{H}$ was used to write Faraday's law in terms of $\mathbf{H}$ rather than $\mathbf{B}$. To be able to substitute this relation into Ampere's law, we first take the curl on both sides of Ampere's law:

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J} + \nabla \times \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{J} + \varepsilon \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

Two conditions are implicit here: that the curl and time derivatives are independent and that permittivity is constant in space. The first is always correct. The second is an assumption and does not have to hold in all situations. As long as the materials are homogeneous, we should have no difficulty with this assumption. The left-hand side can now be written as $\nabla \times (\nabla \times \mathbf{H}) = -\nabla^2\mathbf{H} + \nabla(\nabla \cdot \mathbf{H})$. The term $\nabla \times \mathbf{E}$ from Faraday's law is now substituted into Ampere's law

$$-\nabla^2\mathbf{H} + \nabla(\nabla \cdot \mathbf{H}) = \nabla \times \mathbf{J} - \varepsilon \frac{\partial}{\partial t}\left(\mu \frac{\partial \mathbf{H}}{\partial t}\right)$$

From Maxwell's fourth equation **[Eq. (11.8)]**, assuming μ is also constant in space (material homogeneity condition), $\nabla \cdot \mathbf{H} = \mathbf{0}$. Thus, we get

$$\nabla^2\mathbf{H} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J} \quad \text{(nonhomogeneous)}$$

$$\nabla^2\mathbf{H} - \varepsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \text{(homogeneous)}$$

Note that this wave equation is identical in form to the wave equation for the electric scalar potential in **Example 12.3**. Also, note that if we need the homogeneous wave equation only, it is best to start with the source-free Maxwell's equations.

Exercise 12.2 Obtain the homogeneous wave equation in terms of the electric flux density **D**.

Answer $\nabla^2\mathbf{D} - \mu\varepsilon \dfrac{\partial^2 \mathbf{D}}{\partial t^2} = 0.$

12.3.2 Time-Harmonic Wave Equations

The time-harmonic wave equation is obtained either by starting with the time-harmonic Maxwell's equations and following steps similar to those in the previous section, or with the time-dependent equation and then transforming the resulting time-dependent wave equations to time-harmonic wave equations.

If we choose the latter approach, we simply replace $\partial/\partial t$ by $j\omega$. For example, the time-dependent wave equation in **Eq. (12.6)** can be written in the time-harmonic form as

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} = \mu\varepsilon(j\omega)^2\mathbf{A} = -\omega^2\mu\varepsilon\mathbf{A} \tag{12.7}$$

However, in doing so, we also implicitly changed the variable **A** from a real variable to a phasor even though the same notation is used. The term $e^{j\omega t}$ is implicit in **A**.

We will show next how to obtain a wave equation in terms of the electric field intensity **E**, starting from the time-harmonic Maxwell equations, and how to obtain the wave equation for **H** by transforming the time-dependent wave equation in **Example 12.4**.

To obtain the time-harmonic wave equation in terms of the electric field intensity **E**, we start with Maxwell's equations in time-harmonic form [see **Eqs. (11.68)** through **(11.71)**], but written in terms of **E** and **H**. Assuming linear, isotropic, homogeneous materials, these are

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\mu\mathbf{H} \tag{12.8}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{12.9}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = \mathbf{J} + j\omega\varepsilon\mathbf{E} \tag{12.10}$$

$$\nabla \cdot \varepsilon\mathbf{E} = \rho_v \tag{12.11}$$

We start by taking the curl on both sides of Faraday's law [**Eq. (12.8)**]:

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\nabla \times \mathbf{H}) \tag{12.12}$$

Substituting for $\nabla \times \mathbf{H}$ from Ampere's law **[Eq. (12.10)]**

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\mathbf{J} + j\omega\varepsilon\mathbf{E}) \tag{12.13}$$

Again using the identity $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$

$$-\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -j\omega\mu(\mathbf{J} + j\omega\varepsilon\mathbf{E}) \tag{12.14}$$

The divergence of **E** is given in **Eq. (12.11)**. Separating the current density into source and induced current densities (i.e., $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e = \mathbf{J}_s + \sigma\mathbf{E}$ [A/m^2], where $\mathbf{J}_s$ [A/m^2] is an applied source current density and $\mathbf{J}_e = \sigma\mathbf{E}$ [A/m^2] is an induced current density), substituting, and rearranging terms gives

$$\nabla^2\mathbf{E} = \nabla\left(\frac{\rho_v}{\varepsilon}\right) + j\omega\mu\mathbf{J}_S + j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E}) \tag{12.15}$$

There are three sources of the electric field: One is due to charge distribution in space in the form of the gradient of the charge density [first term on the right-hand side in **Eq. (12.15)**]. The second is due to applied current densities [second term on the right-hand side of **Eq. (12.15)**]. These are external, applied sources. In addition, the time derivative of **B** generates induced current densities as required from Faraday's law. These current densities are represented by the term $\sigma\mathbf{E}$ and are not externally applied. The last term on the right-hand side is due to displacement current densities. The source-free wave equation is obtained if the external sources ρ_v and $\mathbf{J}_s$ are eliminated:

$$\boxed{\nabla^2\mathbf{E} = j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E})} \tag{12.16}$$

If, in addition, the losses are zero ($\sigma = 0$), the source-free, lossless wave equation is obtained:

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon)\mathbf{E} \tag{12.17}$$

Multiplying the terms on the right-hand side and rearranging gives

$$\boxed{\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0} \tag{12.18}$$

This equation is a source-free wave equation in lossless media. It is a commonly used form of the wave equation and forms the basis of the remaining chapters of this book. **Equation (12.18)** is called the ***Helmholtz equation*** for the electric field intensity in lossless media. In lossy media, we use **Eq. (12.16)**.

As mentioned earlier, the time-harmonic wave equation may be obtained from the time-dependent wave equation through the phasor transformation. As an example, consider the source-free wave equation in terms of **H**, obtained in **Example 12.4**. If we replace $\partial/\partial t$ by $j\omega$, we get the time-harmonic, source-free, lossless wave equation for **H**:

$$\boxed{\nabla^2\mathbf{H} + \omega^2\mu\varepsilon\mathbf{H} = 0} \tag{12.19}$$

This equation is identical in form to **Eq. (12.18)** (it is a Helmholtz equation in terms of **H**) and, therefore, must also have an identical form of solution. Both **Eqs. (12.18)** and **(12.19)** are extremely important in electromagnetics, as we shall see shortly.

Exercise 12.3 Find the nonhomogeneous, time-harmonic wave equation in terms of the magnetic flux density **B** in lossless media.

Answer $\nabla^2\mathbf{B} + \omega^2\mu\varepsilon\mathbf{B} + \mu(\nabla \times \mathbf{J}) = 0$.

Exercise 12.4 Find the time-harmonic, source-free, wave equation in terms of the electric scalar potential in lossless media.

Answer $\nabla^2 V + \omega^2\mu\varepsilon V = 0$.

12.3.3 Solution of the Wave Equation

Now that various wave equations have been obtained, it is time to solve them. First, we must decide which wave equation to solve and under what conditions. In principle, it does not matter if we solve one wave equation or another, but, in practice, it is important to solve for the electric and magnetic fields in the domain of interest rather than, say, for the electric scalar potential, since these will be more useful in subsequent discussions. Therefore, we will solve first for the electric field intensity **E** and the magnetic field intensity **H** in lossless media and in the absence of sources. The starting point is **Eq. (12.18)** or **(12.19)**. To observe the behavior of fields and define the important aspects of propagation, we use a one-dimensional wave equation; that is, we assume that the electric field intensity **E** or the magnetic field intensity **H** has a single component in space. The conditions under which we solve the equations are:

(1) Fields are time harmonic.
(2) The electric field intensity is directed in the x direction but varies in the z direction; that is, the field is perpendicular to the direction of propagation.
(3) The medium in which the wave propagates is lossless ($\sigma = 0$).
(4) The wave equation is source free ($\mathbf{J}_s = 0$, $\rho_v = 0$).

This set of assumptions seems to be rather restrictive. In fact, it is not. Although the direction in space is fixed, we are free to choose this direction and we can repeat the solution with a field in any other direction in space. Also, and perhaps more importantly, many of the above assumptions are actually satisfied, at least partially in practical waves. For example, if the electric field intensity at the antenna of a receiver is needed, there is no need to take into account the actual current at the transmitting antenna: only the equivalent field in space. Similarly, propagation in general media, although not identical to propagation in lossless media, is quite similar in many cases. The benefit of this approach is in keeping the solution simple while still capturing all important properties of the wave. The alternative is a more general solution but one that is hopelessly complicated.

In fact, the conditions stated in this section specify what is called a uniform plane wave.

12.3.4 Solution for Uniform Plane Waves

A ***uniform plane wave*** is a wave (i.e., a solution to the wave equation) in which the electric and magnetic field intensities are directed in fixed directions in space and are *constant in magnitude and phase on planes perpendicular to the direction of propagation*.

Clearly, for a field to be constant in amplitude and phase on infinite planes, the source must also be infinite in extent. In this sense, a plane wave cannot be generated in practice. However, many practical situations can approximate plane waves to such an extent that plane waves are actually more common that one might think. For example, suppose a satellite in geosynchronous orbit transmits a TV program to Earth. The satellite is at a distance of approximately 36,000 km. For all practical purposes, it looks to us as a point source and the transmission will be at constant amplitude and phase on the surface of a sphere of radius 36,000 km (in reality, satellite communication is in a fairly narrow beam covering only a small section of the sphere, but on this section, the above conditions apply). This is as good an approximation to a plane wave as one can wish. More important, the receiving antenna is of such small size compared with the distances involved that it sees a plane wave. Thus, analysis of the wave as a plane wave is fully justified, even if the distances involved were smaller. You may want to think in the same way about a radio transmitter on the other side of town or a TV station 50 km away. Thus, the use of plane waves is rather useful, quite general, and will restrict our solutions very little while allowing simplification in both discussion and calculation.

12.3.5 The One-Dimensional Wave Equation in Free-Space and Perfect Dielectrics

With the assumptions in **Section 12.3.3**, the electric field intensity is

$$\mathbf{E} = \hat{\mathbf{x}} E_x(z) \quad [\mathrm{V/m}] \tag{12.20}$$

where **E** is a phasor (i.e., $e^{j\omega t}$ is implied). These assumptions imply the following conditions:

$$E_y = E_z = 0 \quad \text{and} \quad \frac{\partial E_*}{\partial x} = \frac{\partial E_*}{\partial y} = 0 \tag{12.21}$$

where * denotes any component of **E**. Substitution of these into **Eq. (12.18)** results in

$$\frac{d^2 E_x}{dz^2} + \omega^2 \mu \varepsilon E_x = 0 \tag{12.22}$$

where the partial derivative was replaced with the ordinary derivative because of the field dependence on z alone. Also, since the electric field is directed in a fixed direction in space, a scalar equation is sufficient. We denote:

$$\boxed{k = \sqrt{\omega^2 \mu \varepsilon} = \omega \sqrt{\mu \varepsilon} \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right]} \tag{12.23}$$

Equation (12.22) is identical in form to **Eq. (12.1)** (except of course, it is written here in the time-harmonic form); therefore, it has the same type of solution. All we need is to find the functions f and g in the general solution in **Eq. (12.3)**. In this case, since **Eq. (12.22)** describes simple harmonic motion, it has a solution

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz} \quad [\mathrm{V/m}] \tag{12.24}$$

where E_0^+ and E_0^- are constants to be determined from the boundary conditions of the problem. The notations (+) and (−) indicate that the first term is a propagating wave in the positive z direction called a ***forward-propagating wave*** and the second a propagating wave in the negative z direction called a ***backward-propagating wave***, as in **Figure 12.4a** (horizontal arrows indicate the direction of propagation; the electric field intensity components are vertical). The amplitudes E_0^+ and E_0^- are real (but they may, in general, be complex) and are arbitrary. This solution can be verified by direct substitution into **Eq. (12.22)**.

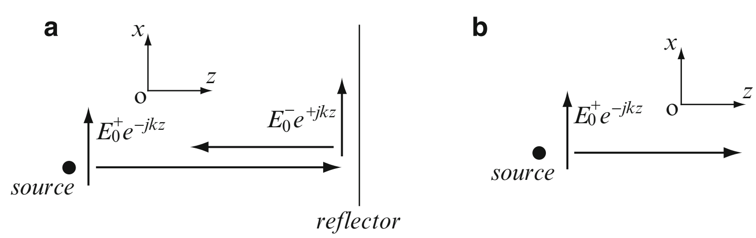

Figure 12.4 (**a**) Forward- and backward-propagating waves in bounded space. (**b**) Forward-propagating wave in unbounded space (the horizontal arrows show the direction of propagation)

Using the phasor transformation, we can write the solution in the time domain as

$$\boxed{E_x(z,t) = \text{Re}\left\{E_x(z)e^{j\omega t}\right\} = E_0^+\cos(\omega t - kz + \phi) + E_0^-\cos(\omega t + kz + \phi) \quad [\text{V/m}]} \tag{12.25}$$

where the initial (arbitrary) phase angle ϕ was added for completeness. Note that this solution is of the same form as the solution in **Eq. (12.3)**.

If the wave propagates in boundless space, only an outward wave exists and E_0^- is zero (all power propagates away from the source and there can be no backward-propagating waves). If the forward-propagating wave is reflected without losses (i.e., for an electric field, this means a perfect conductor; for ripples in the lake, it means a rigid shore), the amplitudes of the two waves are equal. **Figure 12.4b** shows schematically a forward-propagating wave without reflection. Assuming only a forward-propagating wave, the solution is

$$\boxed{E_x(z) = E_0^+ e^{-jkz}e^{j\phi} \quad \text{or} \quad E_x(z,t) = E_0^+\cos(\omega t - kz + \phi) \quad [\text{V/m}]} \tag{12.26}$$

Examining these expressions, it becomes apparent that what changes with time is the phase of the wave. In other words, the phase of the wave "travels" at a certain velocity. To see what this velocity is, we use **Figure 12.5** and follow a fixed point on the wave, for which the phase of the field is $\omega t - kz + \phi =$ constant:

$$z = \frac{\omega t}{k} + \frac{\phi}{k} - \text{constant} \tag{12.27}$$

The speed of propagation of the phase is

$$v_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \quad [\text{m/s}] \tag{12.28}$$

where $k = \omega\sqrt{\mu\varepsilon}$ was used [see **Eq. (12.23)**]. v_p is called the ***phase velocity*** of the wave. If you need a better feel for this velocity, think of a surfer catching a wave. The surfer rides the wave at a fixed point on the wave itself but moves forward at a given velocity. The surfer's velocity is equal to the phase velocity in the case of ocean waves. (The surfer is not moved forward by the wave; rather, the surfer slides down the wave. If it were not for this sliding, only bobbing up and down would occur as the phase of the wave moves forward.) In unbounded, lossless space, the phase velocity and the velocity of the wave or the velocity of transport of energy in the wave are the same. This is not always the case, as we will clearly see in **Chapter 17**. In this chapter, until we start discussing propagation of waves in bounded media, the terms phase velocity and speed or velocity of propagation can be used interchangeably since they happen to be the same. In general, however, they are different. The speed of propagation is a real speed, the speed at which energy propagates (or in the case of an ocean wave, the surfer's speed). The phase velocity is not a real speed in the sense that nothing material moves at that speed; only an imaginary point on the wave moves at this velocity. Because the phase velocity does not relate to physical motion, it can be smaller or larger than the speed of light and, as mentioned, may be different than the velocity of transport of energy.

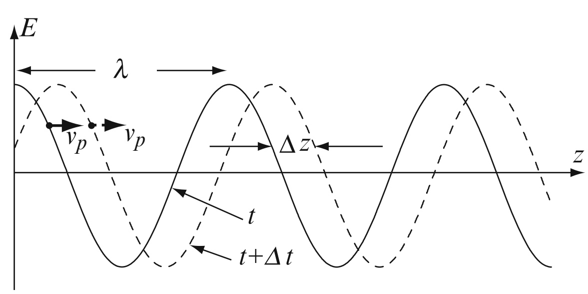

Figure 12.5 Definition of wavelength and calculation of phase velocity

The phase velocity of electromagnetic waves is material dependent. In particular, in free space,

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4 \times \pi \times 10^{-7} \times 8.8541853 \times 10^{-12}}} = 2.997925 \times 10^8 \approx 3 \times 10^8 = c \quad \left[\frac{\text{m}}{\text{s}}\right] \tag{12.29}$$

The phase velocity of electromagnetic waves in free space equals the speed of light. Perhaps this should have been suspected since light is an electromagnetic wave.

The phase velocity in most materials is lower than c since $\mu_r, \varepsilon_r \geq 1$. In fact, the phase velocity in good conductors can be a small fraction of the speed of light.

As the wave propagates, the distance between two successive crests of the wave depends both on the frequency of the wave and its phase velocity. We define the ***wavelength*** λ (in meters) as that distance a wave front (a front of constant phase) travels in one cycle:

$$\lambda = \frac{v_p}{f} = \frac{2\pi v_p}{\omega} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi}{k} \quad [\text{m}] \tag{12.30}$$

The wavelength for the surfer is the distance between two successive crests. This distance is relatively long (perhaps 50 to 100 m). In the electromagnetic case, the wavelength can be very short or very long, depending on frequency and phase velocity. For example, the wavelength in free space for a wave at 50 Hz is 6,000 km. At 30 GHz (a frequency used to communicate with satellites), the wavelength is 10 mm. From the definition of the wavelength in **Eq. (12.30)**, we can write k as

$$k = \frac{2\pi}{\lambda} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.31}$$

k is called the ***wave number***. If the wavelength in free space is given, then k is called the ***free-space wave number***.

Example 12.5 Propagation of Electromagnetic Waves in Water An electromagnetic wave propagates downward from an aircraft and into water. The wave is at a frequency of 10 GHz. Assume that at this frequency distilled water has a relative permittivity of 24 (no losses) and neglect any effect the interface between air and water may have:

(a) Calculate phase velocity, wavelength, and wave number in air.
(b) Calculate phase velocity, wavelength, and wave number in water.
(c) Write the electric field intensities in air and water. Assume the electric and magnetic fields are parallel to the surface of the water, with known but different amplitudes in air and water.

Solution: The phase velocity is calculated from the permeability and permittivity of air and water using **Eq. (12.28)** or **(12.29)**. **Equation (12.26)** is then used to write the electric field since only a forward-propagating wave is assumed to exist. The amplitude of the wave in air and water is generally different. Here, we simply assume E_a is the amplitude in air and E_w is the amplitude in water to indicate this difference. The actual relation between the two amplitudes will be discussed in **Chapter 13**.

(a) The permeability and permittivity in air are μ_0 and ε_0. The phase velocity in air is therefore

$$v_{pa} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4 \times \pi \times 10^{-7} \times 8.854 \times 10^{-12}}} = 2.998 \times 10^8 = c \quad \left[\frac{\text{m}}{\text{s}}\right]$$

The wavelength and wave number in air are

$$\lambda_a = \frac{v_{pa}}{f} = \frac{2.998 \times 10^8}{10^{10}} = 0.03 \quad [\text{m}], \quad k_a = \frac{2\pi}{\lambda_a} = \frac{2\pi}{0.03} = 209.44 \quad \left[\frac{\text{rad}}{\text{m}}\right].$$

(b) In water, $\mu = \mu_0$ and $\varepsilon = 24\varepsilon_0$. The phase velocity, wavelength, and wave numbers are

$$v_{pw} = \frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\varepsilon_r}\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{24 \times 4 \times \pi \times 10^{-7} \times 8.854 \times 10^{-12}}} = \frac{2.998 \times 10^8}{\sqrt{24}} = 6.12 \times 10^7 \quad \left[\frac{\text{m}}{\text{s}}\right],$$

$$\lambda_w = \frac{v_{pw}}{f} = \frac{6.12 \times 10^7}{10^{10}} = 0.00612 \quad [\text{m}],$$

$$k_w = \frac{2\pi}{\lambda_w} = \frac{2\pi}{0.00612} = 1026.04 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

Note that the phase velocity is lower by a factor of $\sqrt{\varepsilon_r} = 4.899$, the wavelength is shorter by a factor of 4.899, and the wave number is 4.899 times larger.

(c) Using **Eq. (12.26)**, we can write the electric fields in water and air. We assume that the normal direction is z and the electric field intensity is in the x or y direction (arbitrarily, but parallel to the surface of the water). The fields in air are

$$E_{air}(z) = E_a e^{-jk_a z} = E_a e^{-j209.44z} \quad \text{or} \quad E_{air}(z,t) = E_a \cos\left(2\pi \times 10^{10} t - 209.44z\right) \quad [\text{V/m}]$$

where we assumed zero initial phase angle and real amplitude. The fields in water are

$$E_{water}(z) = E_w e^{-jk_w z} = E_w e^{-j1026.04z} \quad \text{or} \quad E_{water}(z,t) = E_w \cos\left(2\pi \times 10^{10} t - 1026.04z\right) \quad [\text{V/m}]$$

The differences are in the amplitude and wave number. Since the wave number multiplied by distance z is a phase, the phase of the wave changes much faster in water than in air.

Example 12.6 Suppose a permanent space station is built on Mars and the station communicates regularly with Earth. The distance between Earth and Mars is approximately 100 million km. Calculate the delay between transmission and reception of a signal sent from Mars and received on Earth.

Solution: The intervening medium is free space and, therefore, the speed of propagation is c. The time required for a transmission to reach Earth is

$$t = \frac{d}{v_p} = \frac{100 \times 10^6 \times 10^3}{3 \times 10^8} = 3.33 \times 10^2 = 333 \quad [\text{s}]$$

This is a delay of over 5.5 min. In other words, if a response is required, it cannot be had before 11 min have passed. This also means that timing of any radio-controlled equipment must take into account this delay. Try to imagine the difficulties in communication with distant stars. The nearest star is 3–4 light years away. Any two-way communication will take 6–8 years (if, of course, such vast distances can be covered at all); you better make every word count!

So far, we only discussed the electric field intensity **E**. Maxwell's equations tell us that a magnetic field intensity **H** exists whenever **E** exists. Thus, for a complete discussion of the electromagnetic wave, we must discuss the magnetic field as well. Rather than repeating the process above, we simply substitute the electric field intensity we obtained in Maxwell's first equation **[Eq. (12.8)]** to obtain the magnetic field intensity. The equation in terms of components is

$$\hat{\mathbf{x}}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu\left(\hat{\mathbf{x}}H_x + \hat{\mathbf{y}}H_y + \hat{\mathbf{z}}H_z\right) \tag{12.32}$$

From the assumption that **E** has only an x component, which varies only in z, only the term $\partial E_x/\partial z$ exists. This means that $H_x = H_z = 0$, and **Eq. (12.32)** becomes a scalar equation:

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y \tag{12.33}$$

or writing this for the forward-propagating wave, for H_y,

$$H_y^+(z) = \frac{j}{\omega\mu}\frac{\partial E_x^+}{\partial z} \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{12.34}$$

Calculating the derivative of E_x^+ with respect to z from **Eq. (12.26)**, we get

$$\frac{\partial E_x^+}{\partial z} = \frac{\partial}{\partial z}\left(E_0^+ e^{-jkz}\right) = -jk\left(E_0^+ e^{-jkz}\right) = -jkE_x^+(z) \tag{12.35}$$

Substituting this result in **Eq. (12.32)** gives

$$H_y^+(z) = \frac{k}{\omega\mu}E_x^+(z) \quad \left[\frac{\text{A}}{\text{m}}\right] \tag{12.36}$$

As was mentioned earlier, the reference field is **E** (an arbitrary choice used in electromagnetics as a convention). Thus, we define the ratio between $E_x(z)$ and $H_y(z)$ as

$$\eta = \frac{E_x^+(z)}{H_y^+(z)} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} \quad [\Omega] \tag{12.37}$$

This quantity is an impedance because the electric field intensity is given in [V/m] and the magnetic field intensity is given in [A/m]. The quantity η is called the ***intrinsic impedance*** or ***wave impedance*** of the material since it is only dependent on material properties, as the right-hand side of **Eq. (12.37)** shows. The intrinsic impedance of free space is

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times 10^{-7}}{\frac{1}{36\pi}\times 10^{-9}}} = \sqrt{144\pi^2\times 10^2} = 120\pi = 376.9911184\ \ \Omega \tag{12.38}$$

For application purposes, we use the value of 377 Ω for the intrinsic impedance of free space:

$$\boxed{\eta_0 \cong 377\ \ \Omega} \tag{12.39}$$

The equation for H_y^+ can now be written from **Eq. (12.37)** as

$$\boxed{H_y^+(z) = \frac{1}{\eta}E_x^+(z) \quad \left[\frac{\text{A}}{\text{m}}\right]} \tag{12.40}$$

Note that **H** and **E** propagate in the same direction and are orthogonal to each other and to the direction of propagation. This property makes **E** and **H** ***transverse electromagnetic*** (TEM) waves. The relation between the electric and magnetic fields in space is shown in **Figure 12.6**. This is a very special relation: for an electric field in the positive x direction, the magnetic field must be in the positive y direction (for the wave to propagate in the positive z direction), as the above results indicate. This aspect of the propagation will be defined later in this chapter in terms of the vector product between the electric and magnetic field intensities.

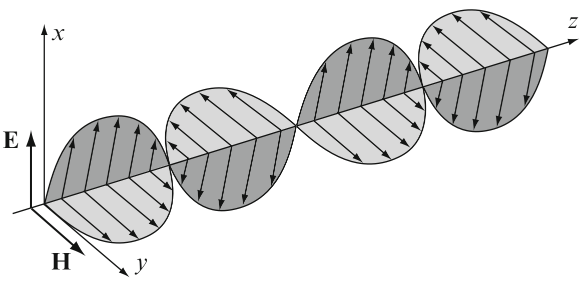

Figure 12.6 The relation between the electric and magnetic field intensities in a plane wave

The above discussion was restricted to a single component of the electric and magnetic field intensities. However, the same can be done with any other component of the electric or magnetic field and any other direction of propagation. The only real restriction on the above properties was the use of the lossless wave equation. This will be relaxed later in this chapter when we discuss propagation of waves in materials.

The properties defined above are important properties of electromagnetic waves. We defined them for time-harmonic uniform plane waves, and, therefore, they are only meaningful for time-harmonic fields. Wavelength and wave number can only properly be defined for time-harmonic fields. On the other hand, phase velocity and intrinsic impedance can be defined in terms of material properties alone and therefore do not depend on the time-harmonic form of the equations.

Example 12.7 An AM radio station transmits at 1 MHz. At some distance from the antenna, the amplitude of the electric field intensity is 10 V/m. The wave propagates from the station outward uniformly in all directions and the electric field intensity is everywhere perpendicular to the direction of propagation. Assume air has the properties of free space:

(a) Find the magnetic field intensity of the wave.
(b) Write the electric and magnetic field intensities in the time domain.
(c) During very heavy rain, the effective relative permittivity of air changes from $\varepsilon_r = 1.0$ to $\varepsilon_r = 1.5$. Calculate the change in phase velocity, intrinsic impedance, and the magnetic field intensity, assuming the amplitude of the electric field intensity remains the same.

Solution: Because the transmission is uniform, propagation is on a spherical surface. At large distances from the source, the spherical surface may be viewed as a plane and, therefore, the transmission may be approximated as a plane wave. **(a)** Assuming that the vertical direction coincides with the z direction, we may write the electric field intensity as z directed and propagating in the x (or, if we wish, in the y) direction, parallel to the surface of the Earth. The magnetic field intensity is then found such that propagation is, indeed, in this direction and away from the station. **(b)** Changes in permittivity affect the intrinsic impedance and therefore the ratio between the electric and magnetic field intensities.

(a) The electric field intensity only varies with x and the wave only propagates outward. Therefore, the electric field intensity has the form

$$\mathbf{E}(x) = \hat{\mathbf{z}} E_0 e^{-jk_0 x} \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

where k_0 is the wave number in free space. To find the magnetic field intensity, we use Faraday's law in Cartesian coordinates:

$$\hat{\mathbf{x}}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu_0\left(\hat{\mathbf{x}} H_x + \hat{\mathbf{y}} H_y + \hat{\mathbf{z}} H_z\right)$$

Since **E** only has a z component and it may only vary with x, only the term $\partial E_z/\partial x$ exists on the left-hand side. On the right-hand side therefore, only the y component may exist. This gives

$$\hat{\mathbf{y}}\frac{\partial E_z}{\partial x} = \hat{\mathbf{y}}\, j\omega\mu_0 H_y$$

The derivative of E_z with respect to x is

$$\frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x}\left[E_0 e^{-jk_0x}\right] = -jk_0E_0e^{-jk_0x}$$

where we assumed the amplitude is independent of x (plane wave). The magnetic field intensity is

$$H_y = \frac{1}{j\omega\mu_0}\frac{\partial E_z}{\partial x} = -\frac{k_0}{\omega\mu_0}E_0e^{-jk_0x} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

In vector form,

$$\mathbf{H}(x) = -\hat{\mathbf{y}}\frac{k_0}{\omega\mu_0}E_0e^{-jk_0x} = -\hat{\mathbf{y}}\frac{E_0}{\eta_0}e^{-jk_0x} \quad \left[\frac{\text{A}}{\text{m}}\right].$$

With the given data:

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{2\pi f}{c} = \frac{2\times\pi\times 10^6}{3\times 10^8} = 0.021 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

and

$$\mathbf{H}(x) = -\hat{\mathbf{y}}\frac{10}{377}e^{-j\,0.021x} = -\hat{\mathbf{y}}\,0.0265\,e^{-j\,0.021x} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

(b) Since we have no other information, we must assume that the initial phase angle is zero. Also, the amplitude of the electric field intensity is known and is a real value. The electric and magnetic field intensities in the time domain are

$$\mathbf{E}(x,t) = \hat{\mathbf{z}}\,\text{Re}\left[E_0e^{-jk_0x}e^{j\omega t}\right] = \hat{\mathbf{z}}E_0\cos(\omega t - k_0x) = \hat{\mathbf{z}}\,10\cos\left(2\pi\times 10^6 t - k_0x\right) \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$\mathbf{H}(x,t) = \hat{\mathbf{y}}\,\text{Re}\left[\frac{-k_0}{\omega\mu_0}E_0e^{-jk_0x}e^{j\omega t}\right] = -\hat{\mathbf{y}}\frac{k_0}{\omega\mu_0}E_0\cos(\omega t - k_0x) = -\hat{\mathbf{y}}\frac{E_0}{\eta_0}\cos(\omega t - k_0x) \quad \left[\frac{\text{A}}{\text{m}}\right]$$

With $\eta_0 = 377\ \Omega$, we get

$$\mathbf{H}(x,t) = -\hat{\mathbf{y}}\,0.0265\cos\left(2\pi\times 10^6 t - 0.021x\right) \quad \left[\frac{\text{A}}{\text{m}}\right].$$

(c) The phase velocity and intrinsic impedance in air (free space) are $v_p = 3\times 10^8$ m/s and $\eta_0 = 377\ \Omega$. The phase velocity in heavy rain is reduced by a factor of $\sqrt{\varepsilon_r} = \sqrt{1.5} = 1.2247$. Thus, in heavy rain the phase velocity is $v_p = 2.4495\times 10^8$ m/s. The intrinsic impedance is given in **Eq. (12.37)**:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{377}{\sqrt{1.5}} = 307.8 \quad [\Omega]$$

Note also, that k increases by the same factor. Thus, the electric and magnetic field intensities become

$$\mathbf{E}(x,t) = \hat{\mathbf{z}}\,10\cos\left(2\pi\times 10^6 t - 0.021\sqrt{1.5}x\right) \quad \left[\frac{\text{V}}{\text{m}}\right]$$

$$\mathbf{H}(x,t) = -\hat{\mathbf{y}}\,0.0325\cos\left(2\pi\times10^6 t - 0.021\sqrt{1.5}x\right)\quad\left[\frac{\text{A}}{\text{m}}\right]$$

The wave number (as well as the wavelength) has changed. The wave number increases, whereas the wavelength decreases. Because the amplitude of the electric field intensity remains the same, the amplitude of the magnetic field intensity has increased by a factor of $\sqrt{1.5}$.

Note: We arbitrarily assigned x as the direction of propagation. Similar results can be obtained by rotating the system of coordinates so that the direction of propagation coincides with any other axis.

Example 12.8 A radar installation transmits a wave whose magnetic field intensity is

$$\mathbf{H} = \hat{\mathbf{x}}H_0\cos(\omega t - k_0 z)\quad\left[\frac{\text{A}}{\text{m}}\right]$$

where $H_0 = 25$ A/m and $f = 30$ GHz. Propagation is in free space and z is the vertical direction. Assume plane waves and lossless propagation. Calculate:

(a) The wave number for the wave.
(b) The electric field intensity of the wave in phasor form.

Solution: The free-space wave number is calculated from the intrinsic impedance of free space which is known. With the intrinsic impedance, we can calculate the magnetic field intensity, using Faraday's law in Cartesian coordinates:

(a) From **Eq. (12.37)**, we write

$$\eta_0 = \frac{\omega\mu_0}{k_0}\quad\rightarrow\quad k_0 = \frac{\omega\mu_0}{\eta_0} = \frac{2\times\pi\times3\times10^{10}\times4\times\pi\times10^{-7}}{377} = 628.3\quad\left[\frac{\text{rad}}{\text{m}}\right]$$

(b) The magnitude of the electric field intensity can be written directly from **Eq. (12.40)**:

$$|\mathbf{E}| = \eta_0|\mathbf{H}| = 377\times25 = 9425\quad\left[\frac{\text{V}}{\text{m}}\right]$$

However, to find the direction of the electric field intensity, we must use Ampere's law in lossless media and in the frequency domain, written here in component form:

$$\hat{\mathbf{x}}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\varepsilon_0\left(\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z\right)$$

Because **H** has only a component in the x direction and only varies with z, only the derivative $\partial H_x/\partial z$ is nonzero on the left-hand side. This term is in the y direction; therefore, the right-hand side can only have a y-directed component:

$$\hat{\mathbf{y}}\frac{\partial H_x}{\partial z} = \hat{\mathbf{y}}\,j\omega\varepsilon_0 E_y$$

To calculate the derivative of **H** with respect to z, we write **H** in phasor form:

$$\mathbf{H} = \hat{\mathbf{x}}H_0 e^{-jk_0 z}\quad[\text{A/m}]$$

The electric field intensity is therefore

$$\mathbf{E} = \hat{\mathbf{y}}\frac{1}{j\omega\varepsilon_0}\frac{\partial H_x}{\partial z} = \hat{\mathbf{y}}\frac{-jk_0}{j\omega\varepsilon_0}H_0 e^{-jk_0 z} = -\hat{\mathbf{y}}\frac{k_0}{\omega\varepsilon_0}H_0 e^{-jk_0 z} = -\hat{\mathbf{y}}\frac{628.3}{2\times\pi\times3\times10^{10}\times8.854\times10^{-12}}\times25e^{-j628.3z}$$
$$= -\hat{\mathbf{y}}\,9425e^{-j628.3z}\quad\left[\frac{\text{V}}{\text{m}}\right]$$

The electric field intensity is in the negative y direction. Note that this relation also implies the following:

$$\eta_0 = \frac{k_0}{\omega\varepsilon_0} = \frac{\omega\mu_0}{k_0} \quad [\Omega]$$

That this must be correct can be shown by direct multiplication (see **Exercise 12.5**)

Exercise 12.5 Show that the following two forms of the intrinsic impedance are identical:

$$\eta = \frac{\omega\mu}{k} \quad \text{and} \quad \eta = \frac{k}{\omega\varepsilon} \quad [\Omega]$$

Now that we know how to write the electric field intensity from the magnetic field intensity and vice versa, it is well to return to the beginning of the section and discuss the idea of a plane wave and its generation a bit more. Suppose that we built an infinite sheet of current with a total line current density of J_s [A/m]. **Figure** 12.7 shows how this might be accomplished, at least in principle, by the use of a stack of infinite wires. If the current is an AC current, then a wave is generated which propagates away from the sheet of current. The magnetic field intensity and the electric field intensities are shown in the figure (the latter is calculated from Ampere's law, as in **Example 12.8**). From Ampere's law, the magnetic field intensity (and therefore the electric field intensity) is constant on any plane parallel to the sheet, and since the sheet is perpendicular to the direction of propagation (negative or positive z directions), this constitutes a true plane wave. Of course, because we cannot physically build the current sheet, we cannot obtain a true plane wave in any physical application.

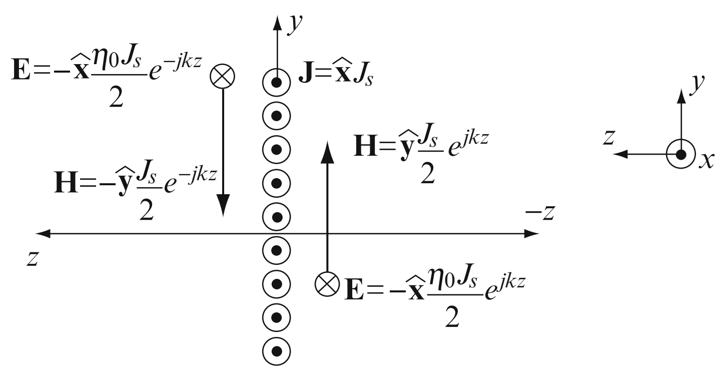

Figure 12.7 Generation of a true uniform plane wave by an infinite current sheet. The plane wave propagates away from the sheet

12.4 The Electromagnetic Spectrum

The previous section alluded to the fact that a low-frequency wave has a long wavelength and a high-frequency wave has a short wavelength, based on **Eq. (**12.30**)**. This fact is quite important in applications of electromagnetic waves. By analogy, we know that low-frequency sound waves propagate to larger distances. Whales use these very low frequencies to communicate. Similarly, a foghorn on a ship produces low-frequency notes. Dolphins, on the other hand, use high-frequency sounds to locate objects in water, as do bats in air. Thus, different portions of the spectrum of sound are used and are useful for different applications. The same applies to the electromagnetic spectrum. You may know, for example, that TV transmission is in the range between 54 MHz and 800 MHz or that most satellite communication occurs between 1,000 MHz (1 GHz) and 30,000 MHz (30 GHz). Similarly your friendly police radar speed detector operates at 10 GHz and 30 GHz. Although we cannot explain at this stage why the various application use (or perhaps require) various frequencies, it must be that some frequency ranges are more appropriate or better suited for some applications.

The electromagnetic spectrum is divided into bands, based either on frequencies or the equivalent wavelength in free space. These bands are, to a large extent, arbitrary and are designated for identification purposes. A graphical representation of the electromagnetic spectrum is shown in **Figure** 12.8. The following should be noted:

(1) The spectrum of electromagnetic waves is between zero and ∞. Although we may not be able to use waves above certain frequencies, they do exist.

(2) Infrared, visible light, ultraviolet rays, X-rays, γ-rays, cosmic rays, etc., are electromagnetic waves.

(3) The narrower bands below the infrared band are arbitrarily divided by wavelength and designated names. Each band is one decade in wavelength.

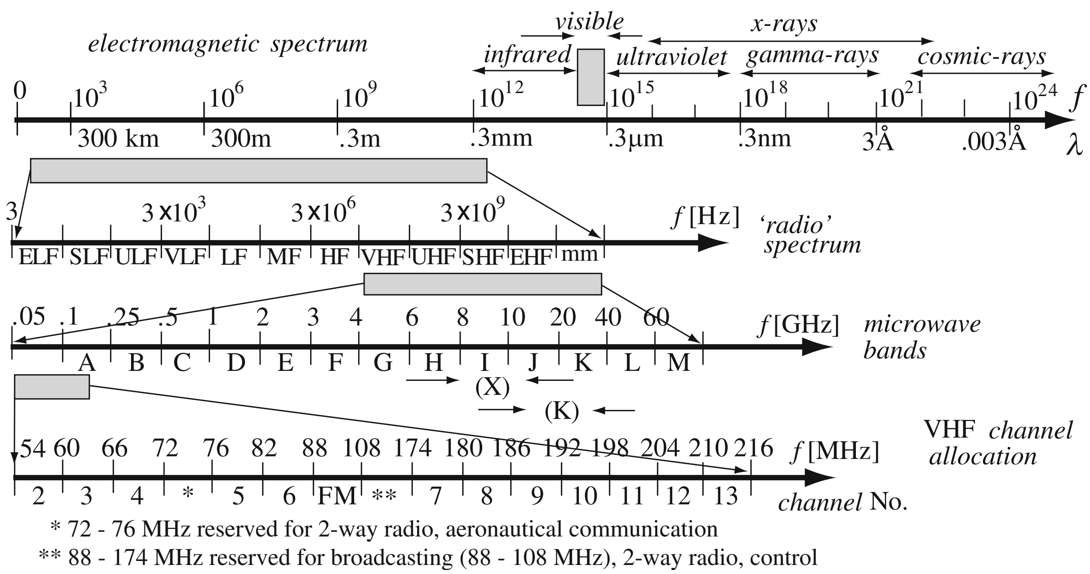

Figure 12.8 The electromagnetic spectrum

Most of our work will have to do with the spectrum below the infrared region since much of the work in light is treated in optics. However, the relations we develop (Snell's law, reflection, transmission, and refraction of waves) apply equally well to higher frequencies. In fact, we will see that some of the relations in optics are simplifications of the electromagnetic relations obtained at lower frequencies.

The electromagnetic spectrum in **Figure 12.8** shows, in addition to frequency ranges, some of the general applications relevant to that range. As mentioned, these bands are arbitrary and the applications do not follow any particular range. For example, RADAR (**R**adio **D**etection **A**nd **R**anging), as used for aircraft detection, guidance, and weather, operates in the SHF and EHF domains. Radar can also be used to detect objects buried underground. Typical frequencies for this application of radar can be as low as 100 MHz, in the VHF band. Similarly, communication with submarines can be done at frequencies below 100 Hz in the SLF band. Sometimes, the frequency used is allocated by convention. In other cases, the frequency or range of frequencies applicable is dictated by the application. For example, one of the frequency ranges allowed for amateur radio (ham radio) is 3.5 to 4 MHz. This is by convention. Other frequency bands may be used as well (and some are), but these have been decided upon so that one group of users does not interfere with another. After all, we may not wish to mix, say, military use of the FM band with civilian FM stations or citizens band radio with air traffic control. Other frequencies such as those for radar or communication with submarines are dictated by the application. In radar equipment, the higher the frequency, the higher the resolution. In communication with satellites, the size of the antennas is dictated by frequency (the higher the frequency, the smaller the antenna). It is therefore of some advantage to use higher frequencies. In communication with submarines, the main effect is that of penetration of waves in water. Low frequencies penetrate well, whereas high frequencies do not. Similarly, microwave ovens operate mostly at 2,450 MHz because at that frequency water absorbs electromagnetic energy and can be heated (microwave ovens heat water; any material that has no water in it, or is not lossy, will not be affected).

The spectrum may be further subdivided for specific purposes. For example, the VHF band may be divided by frequency allocations as shown. Again, this is by convention. Similarly, the microwave region is often divided in bands, each designated with a letter as shown in **Figure 12.8**. In this definition, microwave ovens operate in the E band and police radar detectors operate in the I (previously known as the X band; microwave bands shown in square brackets are old designations shown here for comparison) or K band. With this designation, you can at least get the satisfaction of knowing in which band the radar detector works if you get caught speeding.

12.5 The Poynting Theorem and Electromagnetic Power

One of the most important characteristics of waves is their ability to transport energy and the power associated with the process. Without this ability, many of the most important applications of electromagnetics could not be realized. To examine power and energy relations in the electromagnetic wave, it is convenient to look first at the general time-dependent expression for the rate of energy transfer that includes time rate of change in stored magnetic and stored electric energy and dissipated power. As always, the starting point must be with Maxwell's equations.

Before formalizing the expressions for energy transfer, first consider Ampere's law **[Eq. (11.25)]**:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \left[\frac{\text{A}}{\text{m}^2}\right] \tag{12.41}$$

where **J** includes all possible current densities as follows:

$$\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_e = \mathbf{J}_0 + \sigma \mathbf{E} \quad \left[\text{A/m}^2\right] \tag{12.42}$$

where $\mathbf{J}_0$ indicates source current densities and $\mathbf{J}_e$ indicates induced current densities in conducting media. Now, suppose we take the scalar product of **Eq. (12.41)** with the electric field intensity **E**:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{12.43}$$

In **Chapter 7 [Eq. (7.24)]**, we defined Joule's law as

$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} \quad \left[\frac{\text{W}}{\text{m}^3}\right] \tag{12.44}$$

Thus, the first term on the right-hand side in **Eq. (12.43)** is the volume power density due to current densities. Although we have not yet discussed the meaning of the second term, it is also a volume power density. Both terms on the right-hand side of **Eq. (12.43)** depend only on the electric field intensity **E**. Therefore, these are electric power density terms. We could now integrate **Eq. (12.43)** over a volume to calculate total electric power in the volume. However, a more useful relation is obtained by proceeding with the following vector identity **[Eq. (2.141)]**:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{12.45}$$

The second term on the right-hand side is **Eq. (12.43)** and, therefore, all three terms in **Eq. (12.45)** represent power densities. The first term on the right-hand side results from taking the scalar product of Faraday's law **[Eq. (11.24)]** and the magnetic field intensity **H**:

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{12.46}$$

According to **Eq. (12.45)**, each term in this relation also represents a volume power density, but, now, these are magnetic power densities rather than electric power densities. Using the vector identity in **Eq. (12.45)** and the two relations in **Eqs. (12.43)** and **(12.46)**, we get

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \quad \left[\frac{\text{W}}{\text{m}^3}\right] \tag{12.47}$$

Assuming that we consider the power relations in a volume v, bounded by an area s, the total power in the volume is obtained by integrating over the volume:

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = -\int_v \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\right) dv - \int_v \mathbf{E} \cdot \mathbf{J}\, dv \quad [\text{W}] \tag{12.48}$$

The left-hand side is transformed from a volume integral to a surface integral using the divergence theorem. We also use the following identities:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right),\quad \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\mathbf{H}\cdot\mathbf{B}}{2}\right) \tag{12.49}$$

With these, **Eq. (12.48)** becomes

$$\oint_s(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}=-\frac{\partial}{\partial t}\int_v\left(\frac{\mathbf{H}\cdot\mathbf{B}}{2}+\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right)dv-\int_v\mathbf{E}\cdot\mathbf{J}\,dv\quad[\mathrm{W}] \tag{12.50}$$

or, performing the scalar products

$$\mathbf{H}\cdot\mathbf{B}=\mu\mathbf{H}\cdot\mathbf{H}=\mu H^2,\quad \mathbf{E}\cdot\mathbf{D}=\varepsilon\mathbf{E}\cdot\mathbf{E}=\varepsilon E^2 \tag{12.51}$$

we get

$$\boxed{\oint_s(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}=-\frac{\partial}{\partial t}\int_v\left(\frac{\mu H^2}{2}+\frac{\varepsilon E^2}{2}\right)dv-\int_v\mathbf{E}\cdot\mathbf{J}\,dv\quad[\mathrm{W}]} \tag{12.52}$$

The left-hand side of **Eq. (12.52)** represents the total outward flow of power through the area s bounding the volume v or, alternately, the energy per unit time crossing the surface s. If this flow is inwards, it is a negative flow; if outward, it is positive (because $d\mathbf{s}$ is always positive pointing out of the volume). The expression $\mathbf{E}\times\mathbf{H}$ has units of [V/m] $\times$ [A/m] $=$ W/m^2] and is therefore a surface power density. This power density is called the ***Poynting***[2] ***vector*** $\boldsymbol{\mathcal{P}}$:

$$\boxed{\boldsymbol{\mathcal{P}}=\mathbf{E}\times\mathbf{H}\quad\left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]} \tag{12.53}$$

The advantage of this expression is that it also indicates the direction of the power flow, information that is important for wave propagation calculations. Thus, power flows in the direction perpendicular to both **E** and **H**, according to the right-hand rule we used for the vector product (see **Section 1.3.2**). This will often be used to define or identify the direction of propagation of a wave. If the electric field intensity **E** is known, the magnetic field intensity **H** can always be calculated from the appropriate Maxwell's equation. Then, the direction of propagation of the wave can be found from the vector product of the two.

The first term on the right-hand side of **Eq. (12.52)** represents the time rate of decrease in the potential or stored energy in the system. It has two components: One is the time rate of change of the stored electric energy and the other is time rate of change of the stored magnetic energy.

The second term is due to any sources that may exist in the volume. There are two possibilities that must be considered here. One is that the current density is a source current density such as produced by a battery or a generator inside volume v. The second is a current density provided by external sources (outside the volume v). To understand this, suppose a battery is connected to a resistor as shown in **Figure 12.9**. Note that the current densities in the resistor and in the battery are in the same direction as you would expect from a closed circuit. However, the internal electric field intensity in the battery is opposite the electric field intensity in the resistor. This, again, was discussed in **Chapter 7** and it clearly indicates the difference between a source and a dissipative or load term. When we introduce the term $\mathbf{E}\cdot\mathbf{J}$ in the Poynting theorem, the term will be negative if we introduce the load and positive if we introduce the source. Thus, we distinguish between two situations:

(1) No Sources in the Volume *v*: In this case, all sources are external to the volume, but there may be induced current densities inside the volume. All power in the volume must come from outside sources, whereas in any conducting material, the term **J** can be written as $\mathbf{J}=\sigma\mathbf{E}$. Because all power comes from outside the volume, this case is also called the ***receiver case***. The Poynting theorem now reads

[2] John Henry Poynting (1852–1914) published in 1884 what are now known as the Poynting theorem and the Poynting vector in a paper titled "On the transfer of energy in the electromagnetic field." Poynting also performed extensive experiments aimed to determine the gravitational constant and wrote on radiation and radiation pressure. Although the Poynting vector *is* a pointing vector, remember that pointing is not the same as Poynting.

$$\oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv - \sigma \int_v E^2 dv \quad [\text{W}] \tag{12.54}$$

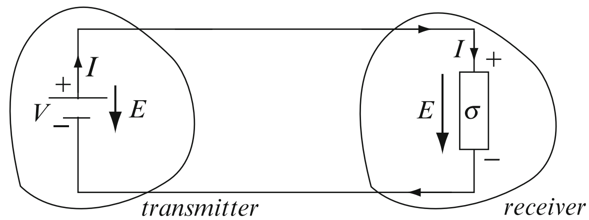

Figure 12.9 Use of a source and load to distinguish between transmitted and received power

Note that both terms on the right-hand side are negative. Thus, the Poynting vector $\mathcal{P} = \mathbf{E} \times \mathbf{H}$ must be negative. This means that power flows into the volume v. For the term $(\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s}$ to be negative, the term $\mathbf{E} \times \mathbf{H}$ must be opposite $d\mathbf{s}$, or into the volume. This situation is shown in **Figure 12.10a**. The receiver shown receives energy from outside its volume. This energy is partly dissipated (in resistive elements) and partly stored in the form of electric and magnetic energy (in inductors and capacitors). Another example is shown in **Figure 12.10b**, which shows food cooking in a microwave oven. The terms of **Eq. (12.54)** now are the rate of decrease in stored electric and magnetic energies in the volume occupied by the food. The third term is that part of the energy flowing into the volume that is converted to heat and does the cooking. Note that the stored energy cannot cook the food: if there are no losses in the food, there will be no energy dissipated and no cooking can take place.

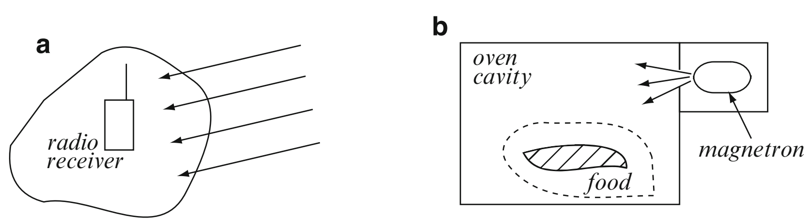

Figure 12.10 (**a**) The receiver case: power enters the volume from outside. (**b**) Example of a receiver case: power enters the food from outside its volume

(2) **Sources in the Volume *v***: The situation discussed here is shown in **Figure 12.11**. A source is located in the volume, but there are no losses in the same volume. Since the electric field intensity **E** and current density **J** are in opposite directions in the source (see **Figure 12.9**), the product $\mathbf{E} \cdot \mathbf{J}$ is negative. Thus, **Eq. (12.52)** now becomes

$$\oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv + \int_v EJ dv \quad [\text{W}] \tag{12.55}$$

As expected, the flow of power is out of the volume (away from the source); therefore, the term $(\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s}$ is positive. This is also called the ***transmitter case***. Two examples are shown in **Figure 12.11**. The first shows a transmitting antenna such as from a mobile telephone. Power is transmitted out. The second shows the microwave oven again, but the volume v now encloses the magnetron (microwave source tube). The power required for cooking is generated in this volume but must be transferred out to do the cooking. No dissipation should occur in the magnetron (otherwise it will itself be "cooked"). In practice, there is quite a bit of power dissipated in the magnetron, and because of this, it must be cooled.

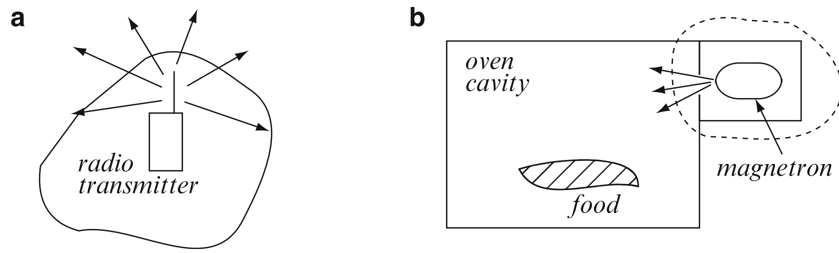

Figure 12.11 (**a**) The general transmitter case: power generated in the volume exits through its surface. (**b**) Power generated in the magnetron is transmitted out into the microwave oven cavity

Of course, both energy sources and dissipative terms may exist in the same volume. In this case, the dissipative term in **Eq. (12.54)** and the source term in **Eq. (12.55)** are present in the equation as two distinct terms. From a practical point of view, dissipative terms may represent losses in generators such as the energy dissipated in the magnetron of the microwave oven. However, for our purposes, it is best to keep the two terms separate.

We chose to work with the Poynting vector using the time-dependent Maxwell's equations. If the electric and magnetic fields do not depend on time, we set the time derivative to zero, in which case only a dissipative term or a source term may exist. There cannot be a rate of change in stored electric or magnetic energy, as we have seen in **Chapter 7**. On the other hand, the direction of the Poynting vector is still valid: it indicates the direction of flow of power (from the source or into a dissipative volume) as we shall see in **Example 12.10**. Also, for electrostatic or magnetostatic applications, there is no dissipation and the Poynting vector shows zero for the very simple reason that the electrostatic field is not accompanied by a magnetic field and the magnetostatic field is not accompanied by an electric field. Thus, the Poynting theorem describes all power relations in a system whether they are electrostatic, magnetostatic, or time dependent. Because the vector product between the electric field intensity and the magnetic field intensity is taken, these two quantities must be related (i.e., they must be generated by the same sources); otherwise the results obtained will have no meaning.

The expressions in **Eqs. (12.52)** through **(12.55)** are instantaneous quantities. For practical purposes, a time-averaged quantity is sometimes more useful. For a periodic time variation of fields, this can be obtained by averaging over a time T (usually a cycle of the field), giving the time-averaged Poynting vector:

$$\boxed{\mathcal{P}_{av} = \frac{1}{T}\int_0^T \mathcal{P}(t)dt \quad \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]} \tag{12.56}$$

The time-averaged Poynting vector is a time-averaged power density. To calculate the total power, either instantaneous or time averaged, the Poynting vector must be integrated over the surface through which the power crosses. This usually means a closed surface enclosing a volume, but not always. The instantaneous power is given as:

$$P(t) = \oint_s \mathcal{P}(t) \cdot d\mathbf{s} = \oint_s (\mathbf{E}(t) \times \mathbf{H}(t)) \cdot d\mathbf{s} \quad [\mathrm{W}] \tag{12.57}$$

whereas the time-averaged power through a closed surface s is

$$P_{av} = \oint_s \mathcal{P}_{av} \cdot d\mathbf{s} \quad [\mathrm{W}] \tag{12.58}$$

There are many cases in which the surface s is an open surface. For example, power may be entering or leaving a volume through a "window." This simply means that the power density outside the window is zero and, therefore, the closed surface integration reduces to integration over the window. Because the Poynting theorem in **Eq. (12.52)** is defined over a closed surface (since it requires the power stored and dissipated in a volume, which is enclosed by a surface s), the relations in **Eqs. (12.57)** and **(12.58)** are written as closed surface integrals.

The important properties of the Poynting theorem and the Poynting vector are as follows:

(1) The Poynting theorem gives the power relations of the fields in any volume.
(2) The Poynting vector is the power density on the surface of a volume. The direction of the Poynting vector is the direction of flow of power.
(3) The Poynting vector gives the direction of propagation of electromagnetic power.
(4) The Poynting theorem gives the net flow of power out of a given volume through its surface.

Example 12.9 Consider a plane wave that generates an electric field intensity $\mathbf{E} = -\hat{\mathbf{y}}E_0\cos(\omega t - kz)$ [V/m], where $E_0 = 1{,}000$ V/m and $f = 300$ MHz. Propagation is in free space:

(a) What is the direction of propagation of the wave?
(b) Calculate the instantaneous and time-averaged power densities in the wave.
(c) Calculate the total instantaneous and time-averaged power carried by the wave.
(d) Suppose a receiving dish antenna is 1 m in diameter. How much power is received by the receiving antenna if the surface of the dish is perpendicular to the direction of propagation of the wave?

Solution: From our discussion on plane waves, the direction of propagation must be in the z direction. However, we can show this from the Poynting vector. For this, we first calculate the magnetic field intensity using Faraday's law. Power density, total power, etc., are all calculated from the Poynting vector:

(a) The magnetic field intensity is found from **Eq. (12.32)** by noting that the electric field intensity has only a y component and varies only with z:

$$-\hat{\mathbf{x}}\frac{\partial E_y}{\partial z} = \hat{\mathbf{x}}j\omega\mu H_x$$

Using the phasor form of **E**

$$\mathbf{E} = -\hat{\mathbf{y}}E_0 e^{-jkz} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

we get

$$\mathbf{H} = \hat{\mathbf{x}}\frac{k}{\omega\mu_0}E_0 e^{-jkz} = \hat{\mathbf{x}}\frac{1}{\eta_0}E_0 e^{-jkz} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

or, in the time domain,

$$\mathbf{H} = \hat{\mathbf{x}}\frac{1}{\eta_0}E_0\cos(\omega t - kz) \quad \left[\frac{\text{A}}{\text{m}}\right]$$

The Poynting vector is

$$\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{H} = [-\hat{\mathbf{y}}E_0\cos(\omega t - kz)]\times\left[\hat{\mathbf{x}}\frac{1}{\eta_0}E_0\cos(\omega t - kz)\right] = \hat{\mathbf{z}}\frac{E_0^2}{\eta_0}\cos^2(\omega t - kz) \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

The direction of flow of power is the z direction. This is also the direction of propagation of the wave.

(b) The instantaneous power density emitted by the antenna is that given in **(a)**:

$$\boldsymbol{\mathcal{P}}(z,t) = \mathbf{E}(z,t)\times\mathbf{H}(z,t) = \hat{\mathbf{z}}\frac{E_0^2}{\eta_0}\cos^2(\omega t - kz) \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

The time-averaged power density is found by integrating the instantaneous power density over one cycle of the wave ($T = 1/f = 2\pi/\omega$):

$$\boldsymbol{\mathcal{P}}_{av}(z) = \hat{\mathbf{z}}\frac{1}{T}\int_0^T\frac{E_0^2}{\eta_0}\cos^2(\omega t - kz)dt = \hat{\mathbf{z}}\frac{1}{T}\frac{E_0^2}{\eta_0}\int_0^T\left[\frac{1}{2}+\frac{1}{2}\cos 2(\omega t - kz)\right]dt$$

$$= \hat{\mathbf{z}}\frac{1}{T}\frac{E_0^2}{\eta_0}\int_0^T\frac{dt}{2} + \hat{\mathbf{z}}\frac{1}{T}\frac{E_0^2}{\eta_0}\int_0^T\left[\frac{1}{2}\cos 2(\omega t - kz)\right]dt \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

The second integral is zero and the first equals $T/2$. The time-averaged power density is therefore

$$\boldsymbol{\mathcal{P}}_{av}(z) = \hat{\mathbf{z}}\frac{E_0^2}{2\eta_0} = \hat{\mathbf{z}}\frac{1000^2}{2\times 377} = \hat{\mathbf{z}}\,1326.26 \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

(c) The power density is uniform throughout space and does not depend on location(except for phase, which varies in the z direction). Thus, both the total instantaneous and time-averaged power are infinite. This is true of any plane wave.

(d) The power received by the antenna equals the power density multiplied by the surface of the antenna. Thus, for a dish of diameter $d = 1$ m, the instantaneous power received is

$$P(t) = \left|\boldsymbol{\mathcal{P}}(z,t)\right|S = \frac{E_0^2\pi d^2}{4\eta_0}\cos^2(\omega t - kz) = \frac{1000^2\times\pi\times 1}{4\times 377}\cos^2\left(6\pi\times 10^8 t - kz\right)$$
$$= 2083.28\cos^2\left(6\pi\times 10^8 t - kz\right) \quad [\text{W}]$$

The time-averaged power received is

$$P_{av} = |\boldsymbol{\mathcal{P}}_{av}|S = \frac{E_0^2\pi d^2}{8\eta_0} = 1326.26\times\pi\times(0.5)^2 = 1041.61 \quad [\text{W}]$$

Example 12.10 A cylindrical conductor of radius a [m] is made of a material with conductivity σ [S/m] and carries a direct current I [A]. Calculate the power loss for a segment of the conductor L [m] long. The conductor is shown in **Figure 12.12**.

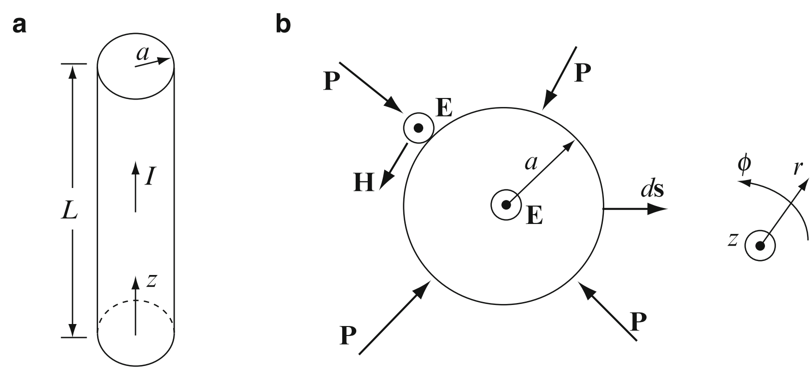

Figure 12.12 (**a**) A segment of a current-carrying conductor. (**b**) Cross section of the conductor viewed from the top of **Figure 12.12a**. The electric and magnetic field intensities and the Poynting vector are shown

Solution: This problem can be solved most easily using the methods of **Chapter 7**. In particular, the resistance of the conductor may be calculated directly followed by calculation of losses using Joule's law. Instead, we will use the Poynting theorem (using the receiver case) to calculate the losses and in the process gain some insight into the loss process. We use **Eq. (12.54)** and evaluate the left-hand side directly. In this case, the time derivatives in **Eq. (12.54)** are zero. The Poynting theorem is therefore

$$\oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = -\sigma\int_v E^2 dv$$

We could calculate the right-hand side, which is exactly Joule's law. Instead, we evaluate the left-hand side. The current density in the conductor is uniform and equal to $\mathbf{J} = \hat{\mathbf{z}}I/S$, where $S = \pi a^2$. In the conductor, $\mathbf{J} = \sigma\mathbf{E}$ and is directed in the z direction. Therefore

$$\mathbf{E} = \hat{\mathbf{z}}\frac{I}{\sigma\pi a^2} \quad \left[\frac{\text{V}}{\text{m}}\right]$$

The magnetic field intensity **H** at the surface of the conductor may be calculated from Ampere's law. Taking a contour around the conductor at $r = a$ and using the right-hand rule, we have (see **Example 8.7**)

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = I \quad \to \quad \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{2\pi a} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

Note that we chose the contour at $r = a$ so that all the conducting material is enclosed by this contour. The power density at the surface of the conductor is

$$\boldsymbol{\mathcal{P}} = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{z}} \frac{I}{\sigma \pi a^2} \times \hat{\boldsymbol{\phi}} \frac{I}{2\pi a} = -\hat{\mathbf{r}} \frac{I^2}{2\sigma \pi^2 a^3} \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

This is the power density entering the conductor through its outer surface. Note that for a given current, the dissipated power density is inversely proportional to conductivity.

The most interesting aspect of this calculation is that this power is directed into the conductor, which means that it must be dissipated in the conductor. The total power is the integral of this power density over the entire surface of the conductor. In the case of the cylinder of length L, the surface is made of the cylindrical surface and the two bases. Thus, we evaluate the total power P from **Eq. (12.54)**. In this case, all time derivatives are zero so there is no change in the stored electric or magnetic energy:

$$P = \oint_s \boldsymbol{\mathcal{P}} \cdot d\mathbf{s} = \left(\int_{cyl} \boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{cyl} + \int_{lb} \boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{lb} + \int_{ub} \boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{ub} \right) \quad [\text{W}]$$

where $cyl =$ cylindrical surface, $lb =$ lower base, and $ub =$ upper base. The three surface vectors are $d\mathbf{s}_{cyl} = \hat{\mathbf{r}}\, a\, d\phi dz$, $d\mathbf{s}_{lb} = -\hat{\mathbf{z}}\, ds$, and $d\mathbf{s}_{ub} = \hat{\mathbf{z}}\, ds$. The last two integrals vanish (because $d\mathbf{s}_{lb}$ and $d\mathbf{s}_{ub}$ are perpendicular to $\hat{\mathbf{r}}$, the scalar products $\boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{ub}$ and $\boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{lb}$ are zero), and we can write

$$P = \int_{cyl} \boldsymbol{\mathcal{P}} \cdot d\mathbf{s}_{cyl} = -\int_{cyl} \left(\hat{\mathbf{r}} \frac{I^2}{2\sigma \pi^2 a^3} \right) \cdot \hat{\mathbf{r}}\, a\, d\phi\, dz = -\int_{\phi=0}^{\phi=2\pi} \left[\int_{z=0}^{z=L} \frac{I^2 a\, dz}{2\sigma \pi^2 a^3} \right] d\phi = -\frac{I^2 L}{\sigma \pi a^2} \quad [\text{W}]$$

This power is negative indicating losses. Also, from the calculation of resistance in **Chapter 7**, we know the resistance of a conductor of length L is $L/\sigma S$, where $S = \pi a^2$. Thus, not surprisingly, this result is the same as that obtained from Joule's law with the exception of the negative sign:

$$P = I^2 R = \frac{I^2 L}{\sigma \pi a^2} \quad [\text{W}]$$

As expected from **Eq. (12.54)**, this is the negative of the power obtained using the Poynting vector. It demonstrates that the dissipated power in the conductor (due to resistance) can be obtained through the power penetrating into the conductor by the electromagnetic fields. By use of the Poynting vector, the power dissipated in the conductor does not enter the conductor through its connections but through the electric and magnetic fields generated by the power source (i.e., a battery) and penetrates through the surface of the conductor. Thus, unlike the common view of current flowing through the conductor and encountering resistance, the use of the Poynting vector indicates that power is propagated by the electric and magnetic fields. In effect, the conductor is not necessary for the propagation of power but is used to guide the power where it is needed. Dissipation is a consequence of the conductors not being ideal.

Example 12.11 Time-Averaged Power Density in Sinusoidal Fields Consider an electric field intensity and a magnetic field intensity generated by a time-harmonic source as $\mathbf{E} = \mathbf{E}_p e^{j\omega t}$ [V/m] and $\mathbf{H} = \mathbf{H}_p e^{j\omega t}$ [A/m], where **E** and **H** are phasors. $\mathbf{E}_p$ and $\mathbf{H}_p$ are complex, given as $\mathbf{E}_p = \mathbf{E}_r + j\mathbf{E}_i$ [V/m] and $\mathbf{H}_p = \mathbf{H}_r + j\mathbf{H}_i$ [A/m]:

(continued)

Example 12.11 (continued)

(a) Calculate the time-averaged Poynting vector.
(b) Using the properties of complex numbers, show that the time-averaged Poynting vector may be written as

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E}\times\mathbf{H}^*\} \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

where * indicates the complex conjugate.

Solution: We first write the time-dependent form of the electric and magnetic field intensities and then use these to calculate the time-dependent Poynting vector using **Eq. (12.53)**. The time-averaged Poynting vector is then obtained using **Eq. (12.56)**, where T is the time of one cycle $(T = 2\pi/\omega)$:

(a) The time domain form of the electric field intensity is written from the definition of phasors as

$$\mathbf{E}(t) = \text{Re}\{(\mathbf{E}_r + j\mathbf{E}_i)e^{j\omega t}\} = \text{Re}\{(\mathbf{E}_r + j\mathbf{E}_i)(\cos\omega t + j\sin\omega t)\} = \mathbf{E}_r\cos\omega t - \mathbf{E}_i\sin\omega t \quad [\text{V/m}]$$

and similarly for the magnetic field intensity

$$\mathbf{H}(t) = \mathbf{H}_r\cos\omega t - \mathbf{H}_i\sin\omega t \quad [\text{A/m}]$$

Now, the time-dependent Poynting vector may be written as

$$\begin{aligned}\mathcal{P}(t) &= \mathbf{E}(t)\times\mathbf{H}(t) = (\mathbf{E}_r\cos\omega t - \mathbf{E}_i\sin\omega t)\times(\mathbf{H}_r\cos\omega t - \mathbf{H}_i\sin\omega t)\\ &= \mathbf{E}_r\times\mathbf{H}_r\cos^2\omega t + \mathbf{E}_i\times\mathbf{H}_i\sin^2\omega t - \mathbf{E}_i\times\mathbf{H}_r\sin\omega t\cos\omega t - \mathbf{E}_r\times\mathbf{H}_i\sin\omega t\cos\omega t \quad [\text{W/m}^2]\end{aligned}$$

The time-averaged Poynting vector is calculated from **Eq. (12.56)**:

$$\mathcal{P}_{av} = \frac{1}{T}\int_0^{t=T}\mathbf{E}(t)\times\mathbf{H}(t)dt$$

$$= \mathbf{E}_r\times\mathbf{H}_r\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\cos^2\omega t\,dt + \mathbf{E}_i\times\mathbf{H}_i\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\sin^2\omega t\,dt - (\mathbf{E}_i\times\mathbf{H}_r + \mathbf{E}_r\times\mathbf{H}_i)\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\sin\omega t\cos\omega t\,dt \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

where $T = 2\pi/\omega$ was used. For clarity, we integrate each term separately:

$$\mathbf{E}_r\times\mathbf{H}_r\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\cos^2\omega t\,dt = \mathbf{E}_r\times\mathbf{H}_r\frac{\omega}{2\pi}\left[\frac{t}{2}+\frac{\sin2\omega t}{4\omega}\right]_{t=0}^{t=2\pi/\omega} = \frac{\mathbf{E}_r\times\mathbf{H}_r}{2}$$

$$\mathbf{E}_i\times\mathbf{H}_i\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\sin^2\omega t\,dt = \mathbf{E}_i\times\mathbf{H}_i\frac{\omega}{2\pi}\left[\frac{t}{2}+\frac{\sin2\omega t}{4\omega}\right]_{t=0}^{t=2\pi/\omega} = \frac{\mathbf{E}_i\times\mathbf{H}_i}{2}$$

$$(\mathbf{E}_i\times\mathbf{H}_r + \mathbf{E}_r\times\mathbf{H}_i)\frac{\omega}{2\pi}\int_0^{t=2\pi/\omega}\sin\omega t\cos\omega t\,dt = (\mathbf{E}_i\times\mathbf{H}_r + \mathbf{E}_r\times\mathbf{H}_i)\frac{\omega}{2\pi}\left[\frac{\sin^2\omega t}{2\omega}\right]_{t=0}^{t=2\pi/\omega} = 0$$

Therefore, the time-averaged Poynting vector is

$$\mathcal{P}_{av} = \frac{\mathbf{E}_\text{r}\times\mathbf{H}_\text{r} + \mathbf{E}_\text{i}\times\mathbf{H}_\text{i}}{2} \quad \left[\frac{\text{W}}{\text{m}^2}\right].$$

(b) Starting with the phasor description of the vectors **E** and **H**, we write

$$\mathbf{E} \times \mathbf{H}^* = (\mathbf{E}_r + j\mathbf{E}_i)e^{j\omega t} \times (\mathbf{H}_r - j\mathbf{H}_i)e^{-j\omega t} = \mathbf{E}_r \times \mathbf{H}_r + \mathbf{E}_i \times \mathbf{H}_i + j(\mathbf{E}_i \times \mathbf{H}_r - \mathbf{E}_r \times \mathbf{H}_i)$$

Comparing this with the result in **(a)**, we can write the time-averaged Poynting vector as

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E} \times \mathbf{H}^*\} \quad \left[\frac{\text{W}}{\text{m}^2}\right].$$

12.6 The Complex Poynting Vector

As pointed out earlier, most electromagnetic relations encountered here, including most applications, are handled in the frequency domain, assuming sinusoidal excitation. Thus, it often becomes necessary to define the Poynting vector in the frequency domain. This definition also shows the relation between real and reactive power and is closely related to time-averaged power.

In **Example 12.11**, we calculated the time-averaged Poynting vector in a general field under sinusoidal conditions as

$$\boxed{\mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E} \times \mathbf{H}^*\} \quad \left[\frac{\text{W}}{\text{m}^2}\right]} \tag{12.59}$$

where * indicates the complex conjugate form. Since the fields used to find this relation were completely general phasors, this relation applies for any sinusoidal fields. Comparing this to **Eq. (12.53)**, we are led to define a complex Poynting vector as

$$\boxed{\mathcal{P}_c = \mathbf{E} \times \mathbf{H}^* = \mathbf{E}^* \times \mathbf{H} \quad \left[\frac{\text{W}}{\text{m}^2}\right]} \tag{12.60}$$

where **E** and **H** are phasors. Clearly, the value of the complex Poynting vector is the ease with which the time-averaged power density and, therefore, time-averaged power are evaluated.

A formal derivation of the complex Poynting vector starts with Maxwell's first two equations in the frequency domain **[Eqs. (11.68)** and **(11.69)]**:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{12.61}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\varepsilon\mathbf{E} \tag{12.62}$$

The conjugates of **Eqs. (12.61)** and **(12.62)** are

$$\nabla \times \mathbf{E}^* = j\omega\mu\mathbf{H}^* \tag{12.63}$$

$$\nabla \times \mathbf{H}^* = \mathbf{J}^* - j\omega\varepsilon\mathbf{E}^* \tag{12.64}$$

The current density **J*** includes source and induced current densities [see **Eq. (12.42)]**

$$\mathbf{J}^* = \mathbf{J}_0^* + \mathbf{J}_e^* = \mathbf{J}_0^* + \sigma\mathbf{E}^* \tag{12.65}$$

First we write the scalar product between **H*** and **Eq. (12.61)** as

$$\mathbf{H}^* \cdot (\nabla \times \mathbf{E}) = -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* \tag{12.66}$$

Next we write the scalar product between **E** and **Eq. (12.64)** as

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}^*) = (\mathbf{J}^* - j\omega\varepsilon\mathbf{E}^*) \cdot \mathbf{E} \tag{12.67}$$

Equations (12.66) and (12.67) may be combined using the following vector identity [see **Eq. (2.141)**]:

$$\mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*) = \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) \tag{12.68}$$

Substituting for $\mathbf{H}^* \cdot (\nabla \times \mathbf{E})$ from **Eq.** (12.66) and for $\mathbf{E} \cdot (\nabla \times \mathbf{H}^*)$ from **Eq.** (12.67) and rearranging terms gives

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = j\omega(\varepsilon \mathbf{E} \cdot \mathbf{E}^* - \mu \mathbf{H} \cdot \mathbf{H}^*) - \mathbf{E} \cdot \mathbf{J}^* \tag{12.69}$$

The first two terms on the right-hand side represent the electric and magnetic power densities. The third term represents the input and dissipated power densities.

Using the ideas of the transmitter and receiver cases discussed in the previous section, the term $\mathbf{E} \cdot \mathbf{J}^*$ is replaced by $\sigma \mathbf{E} \cdot \mathbf{E}^*$ for the receiver case and by $-\mathbf{E} \cdot \mathbf{J}^*$ for the transmitter case, as was done earlier. To write this in terms of power rather than power density, we integrate **Eq.** (12.69) over an arbitrary volume v:

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) dv = j\omega \int_v (\varepsilon \mathbf{E} \cdot \mathbf{E}^* - \mu \mathbf{H} \cdot \mathbf{H}^*) dv - \int_v \mathbf{E} \cdot \mathbf{J}_0^* dv - \int_v \sigma \mathbf{E} \cdot \mathbf{E}^* dv \quad [\mathrm{W}] \tag{12.70}$$

Using the divergence theorem on the left-hand side, we get

$$\boxed{\oint_s (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = j\omega \int_v (\varepsilon \mathbf{E} \cdot \mathbf{E}^* - \mu \mathbf{H} \cdot \mathbf{H}^*) dv - \int_v \mathbf{E} \cdot \mathbf{J}_0^* dv - \int_v \sigma \mathbf{E} \cdot \mathbf{E}^* dv \quad [\mathrm{W}]} \tag{12.71}$$

where $\mathbf{J}_0$ indicates a source current density. The left-hand side is the complex power flow through the surface s enclosing the volume v. The first term on the right-hand side is the reactive power in the volume, the second term on the right-hand side is the complex source power (either positive or negative depending on the location of the source), and the last term is the dissipated power in the volume if dissipation occurs (in conducting media).

Equation (12.71) is the complex Poynting theorem. As mentioned at the beginning of this section when using the complex Poynting vector, it is for the purpose of calculating time-averaged quantities. It is therefore more useful to write this relation as two terms as follows [using the notation in **Eq.** (12.59)]:

$$\boxed{\frac{1}{2}\mathrm{Re}\left\{\oint_s \boldsymbol{\mathcal{P}}_c \cdot d\mathbf{s}\right\} = -\frac{1}{2}\int_v \mathbf{E} \cdot \mathbf{J}_0^* dv - \frac{1}{2}\int_v \sigma \mathbf{E} \cdot \mathbf{E}^* dv \quad [\mathrm{W}]} \tag{12.72}$$

$$\boxed{\frac{1}{2}\mathrm{Im}\left\{\oint_s \boldsymbol{\mathcal{P}}_c \cdot d\mathbf{s}\right\} = \omega \int_v \left(\frac{\varepsilon \mathbf{E} \cdot \mathbf{E}^*}{2} - \frac{\mu \mathbf{H} \cdot \mathbf{H}^*}{2}\right) dv \quad [\mathrm{W}]} \tag{12.73}$$

Equation (12.72) gives the real power balance in the volume. The left-hand side is the net outward flow of power through the surface enclosing the volume. The first term on the right-hand side is the net source power (in this case, the source is outside the volume hence the negative sign) and the last term is the dissipated power in the volume. $\mathbf{E} \cdot \mathbf{J}^*$ is positive for the receiver case and negative for the transmitter case as was discussed in **Section** 12.5. Therefore, the second term on the right-hand side of **Eq.** (12.72) is negative for the receiver case and positive for the transmitter case.

Usually, in the transmitter case, we will assume there are no losses in the volume, whereas in the receiver case, there are no sources in the volume. If this is so, the corresponding terms are deleted from **Eqs.** (12.72).

Equation (12.73) is the balance of reactive power. It shows it is the rate of flow of reactive power across the surface. The right-hand side gives the time-averaged reactive power. From the result in **Example 12.11** and **Exercise 12.6**, we can write the stored, time-averaged magnetic and electric energy densities as

$$w_{m(av)} = \frac{1}{4}\mu \mathbf{H} \cdot \mathbf{H}^*, \quad w_{e(av)} = \frac{1}{4}\varepsilon \mathbf{E} \cdot \mathbf{E}^* \quad \left[\frac{\mathrm{J}}{\mathrm{m}^3}\right] \tag{12.74}$$

To emphasize the time-averaged power densities, **Eq.** (12.73) may be written as

$$\boxed{\frac{1}{2}\mathrm{Im}\left\{\oint_s \boldsymbol{\mathcal{P}}_c \cdot d\mathbf{s}\right\} = 2\omega \int_v \left(\frac{\varepsilon \mathbf{E} \cdot \mathbf{E}^*}{4} - \frac{\mu \mathbf{H} \cdot \mathbf{H}^*}{4}\right) dv \quad [\mathrm{W}]} \tag{12.75}$$

Example 12.12 Consider again the magnetic and electric fields obtained in **Example 12.8**, but now these are given in the frequency domain as

$$\mathbf{E} = -\hat{\mathbf{y}}\,\eta_0 H_0 e^{-jkz} \quad \left[\frac{\text{V}}{\text{m}}\right] \quad \text{and} \quad \mathbf{H} = \hat{\mathbf{x}}\,H_0 e^{-jkz} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

where $H_0 = 25$ A/m and frequency is 30 GHz. Propagation is in free space and z is the vertical direction:

(a) Calculate the time-averaged power density in the wave.
(b) Write the stored electric and magnetic energy densities separately.

Solution: The time-averaged Poynting vector is calculated using **Eq. (12.59)** and the stored electric and magnetic energy densities are given in **Eq. (12.74)**:

(a) First, we need to calculate the complex conjugate of **H**. To do so, we note that H_0 is real and the complex conjugate of e^{-jkz} is e^{+jkz}. Thus,

$$\mathbf{H}^* = \hat{\mathbf{x}}\,H_0 e^{jkz} \quad [\text{A/m}]$$

The time-averaged Poynting vector is

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\text{Re}\left(-\hat{\mathbf{y}}\,\eta_0 H_0 e^{-jkz}\times\hat{\mathbf{x}}\,H_0 e^{jkz}\right) = \hat{\mathbf{z}}\,\frac{\eta_0 H_0^2}{2} \quad \left[\frac{\text{W}}{\text{m}^2}\right]$$

Thus, power is transferred in the positive z direction. The time-averaged power density is

$$\mathcal{P}_{av} = \hat{\mathbf{z}}\,\frac{\eta_0 H_0^2}{2} = \hat{\mathbf{z}}\,\frac{377\times 25^2}{2} = \hat{\mathbf{z}}\,117.81\times 10^3 \quad \left[\frac{\text{W}}{\text{m}^2}\right].$$

(b) The time-averaged stored electric and magnetic energy densities are

$$w_m = \frac{1}{4}\mu_0\mathbf{H}\cdot\mathbf{H}^* = \frac{1}{4}\mu_0\left(\hat{\mathbf{x}}\,H_0 e^{-jkz}\right)\cdot\left(\hat{\mathbf{x}}\,H_0 e^{jkz}\right) = \frac{\mu_0 H_0^2}{4} = \frac{4\times\pi\times 10^{-7}\times 25^2}{4} = 1.963\times 10^{-4} \quad \left[\frac{\text{J}}{\text{m}^3}\right]$$

$$w_e = \frac{1}{4}\varepsilon_0\mathbf{E}\cdot\mathbf{E}^* = \frac{\varepsilon_0}{4}\left(-\hat{\mathbf{y}}\,\eta_0 H_0 e^{-jkz}\right)\cdot\left(-\hat{\mathbf{y}}\,\eta_0 H_0 e^{jkz}\right) = \frac{\eta_0^2\varepsilon_0 H_0^2}{4} = \frac{8.854\times 10^{-12}\times 377^2\times 25^2}{4} = 1.963\times 10^{-4} \quad \left[\frac{\text{J}}{\text{m}^3}\right]$$

Note that the stored electric and magnetic energy densities are rather small and are equal in magnitude.

Exercise 12.6 From the results in **Example 12.11**, show that the time-averaged stored electric and magnetic energy densities for time-harmonic fields are

$$w_{m(av)} = \frac{1}{4}\mu\mathbf{H}\cdot\mathbf{H}^*, \quad w_{e(av)} = \frac{1}{4}\varepsilon\mathbf{E}\cdot\mathbf{E}^* \quad \left[\frac{\text{J}}{\text{m}^3}\right].$$

Exercise 12.7 Given a plane electromagnetic wave with electric field intensity

$$\mathbf{E} = \hat{\mathbf{x}}\,E_0 e^{jkz} \quad [\text{V/m}]$$

show that the time-averaged power density at any point in space may be written as

$$\mathcal{P}_{av} = \frac{E_0^2}{2\eta} \quad \left[\frac{\text{W}}{\text{m}^2}\right].$$

12.7 Propagation of Plane Waves in Materials

Waves.m

That waves are affected by the material in which they propagate has been shown in **Section 12.3.5**, where propagation in lossless dielectrics, including free space, was discussed. The phase velocity, wavelength, wave number, and intrinsic impedance are material dependent. We also know from day-to-day experience that different materials affect waves differently. For example, when you pass under a bridge or through a tunnel, your radio ceases to receive. Propagation in water is vastly different than propagation in free space. If you ever listened to a shortwave radio, you experienced much better reception during the night than during the day. All these are due to effects of materials or environmental conditions on waves.

This aspect of propagation of waves is discussed next because it is extremely important both to understanding of propagation and to applications of electromagnetic waves. Based on the propagation properties of waves, we can choose the appropriate frequencies, type of wave, power, and other parameters needed for design.

In the process, we define the important parameters of propagating waves which, in addition to those defined in **Section 12.3**, describe an electromagnetic wave. These parameters include the propagation, phase, and attenuation constants, as well as the skin depth and the complex permittivity. These parameters will then be used for the remainder of the book to describe the behavior of waves in a number of important configurations, including transmission lines, waveguides, and antennas.

12.7.1 Propagation of Plane Waves in Lossy Dielectrics

A lossy dielectric is a material which, in addition to polarization of charges, conducts free charges to some extent. In simple terms, it is a poor insulator, whereas a perfect dielectric is a perfect insulator. For our purpose, a lossy dielectric is characterized by its permittivity and conductivity. Thus, we may assume that in addition to displacement currents, there are also conduction currents in the dielectric. The assumption that there are no sources in the solution domain is still valid.

The source-free wave equation with losses was written in **Eq. (12.16)** for the electric field intensity as

$$\nabla^2\mathbf{E} = j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E} \tag{12.76}$$

Compare this with the lossless equation ($\sigma = 0$)

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon)\mathbf{E} \tag{12.77}$$

Note that the two equations are of exactly the same form if the term $\sigma + j\omega\varepsilon$ in **Eq. (12.76)** is replaced with a complex term $j\omega\varepsilon_c$; that is,

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon \tag{12.78}$$

The term ε_c can be written as

$$\varepsilon_c = \frac{\sigma + j\omega\varepsilon}{j\omega} = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right] = \varepsilon + j\varepsilon'' \quad \left[\frac{\mathrm{F}}{\mathrm{m}}\right] \tag{12.79}$$

This is called the ***complex permittivity*** and, in general, replaces the permittivity ε in the field equations. The imaginary part of the complex permittivity is associated with losses. Now, the term lossless dielectric becomes obvious: these are dielectrics in which $\sigma = 0$ and ε_c is real and equal to ε. The definition of complex permittivity is not merely a mathematical nicety: it is an accurate model of material behavior. The real and imaginary parts of the complex permittivity are measurable.

The ratio between the imaginary and real parts of the complex permittivity is called the ***loss tangent***[3] of the material and is a common measure of how lossy materials are:

$$\tan\theta_{loss} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'} \quad [\text{dimensionless}] \tag{12.80}$$

Since the loss tangent may be viewed as the ratio between induced and displacement current densities we will use it to define approximation limits to the complex permittivity. A very low conductivity means that the permittivity is real, whereas a high conductivity means that the imaginary part of the complex permittivity dominates and the real part may be neglected.

To obtain a solution to the wave equation in lossy media, we will rely on the solution we already obtained for the lossless equation. Since the two are identical in form if the permittivity in the lossless equation is replaced with the complex permittivity, we can write the wave equation in lossy dielectrics as

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon_c)\mathbf{E} = j\omega\mu\left(j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\right)\mathbf{E} \tag{12.81}$$

or, writing this in the form of the Helmholtz equation in **Eq. (12.18)**,

$$\nabla^2\mathbf{E} - j\omega\mu\left(j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\right)\mathbf{E} = 0 \tag{12.82}$$

Comparing this with the source-free (Helmholtz) equation and denoting

$$\boxed{\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]}} \tag{12.83}$$

Equation (12.82) can be written as

$$\boxed{\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0} \tag{12.84}$$

The quantity γ is called the ***propagation constant*** and is, in general, a complex number. The propagation constant can also be written directly from **Eq. (12.76)** by comparison with **Eq. (12.84)** as

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \tag{12.85}$$

Equation (12.84) for lossy materials is similar to **Eq. (12.18)** for lossless materials. We can put them in exactly the same form if we write

$$\gamma = jk_c \tag{12.86}$$

where

$$k_c = \omega\sqrt{\mu\varepsilon}\sqrt{\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.87}$$

The importance of this is that now we can use all the relations obtained for the lossless propagation of waves by replacing the term jk in **Eqs. (12.24)** and **(12.26)** by γ.

The general solution for propagation in a lossy dielectric has the same two wave components as in **Eq. (12.24)**: one traveling in the positive z direction, the other in the negative z direction

$$E_x(z) = E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z} \quad [\text{V/m}] \tag{12.88}$$

[3] A more accurate description of complex permittivity includes also polarization losses which are due to friction between molecules in the dielectric. These losses add to the real part in **Eq. (12.78)** and therefore to the loss tangent. Our view here is that polarization losses add to conductivity, making σ a total effective conductivity. We will use the loss tangent in **Sections 12.7.2** and **12.7.3** for the sole purpose of defining limits of approximation for the complex permittivity.

Similarly, assuming only an outgoing wave, we have from **Eq. (12.26)**

$$E_x(z) = E_0^+ e^{-\gamma z} \quad [\mathrm{V/m}] \tag{12.89}$$

Since the propagation constant is a complex number (see **Exercise 12.8**), it can also be written as

$$\boxed{\gamma = \alpha + j\beta} \tag{12.90}$$

This gives for the general solution

$$E_x(z) = E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{+\alpha z} e^{+j\beta z} \quad [\mathrm{V/m}] \tag{12.91}$$

Similarly, in the case of forward propagation only

$$E_x(z) = E_0^+ e^{-(\alpha+j\beta)z} = E_0^+ e^{-\alpha z} e^{-j\beta z} \quad [\mathrm{V/m}] \tag{12.92}$$

The general solution in the time domain may be written as

$$E_x(z,t) = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + E_0^- e^{+\alpha z}\cos(\omega t + \beta z) \quad [\mathrm{V/m}] \tag{12.93}$$

For a wave propagating in the positive z direction only, this reduces to the first term of **Eq. (12.93)**:

$$E_x(z,t) = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z) \quad [\mathrm{V/m}] \tag{12.94}$$

In this form, the propagating wave has the same form as **Eq. (12.26)** where β has replaced k and the exponential term $e^{-\alpha z}$ multiplies the amplitude. This is therefore a wave, propagating in the positive z direction, with a phase velocity v_p and with an exponentially decaying amplitude. Thus, unlike the lossless case, in which the amplitude remained constant as the wave propagated, this time the amplitude changes as the wave propagates (**Figure 12.13**). Much more will be said about this decay in amplitude in future sections and chapters. Perhaps the most important general comment is that the decay can be quite rapid and that it depends on conductivity. If $\sigma = 0$, $\alpha = 0$, $e^{-\alpha z} = 1$, and the amplitude does not decay as the wave propagates.

For now, we simply note that α causes an attenuation of the amplitude of the wave and is called the ***attenuation constant***. The attenuation constant α is measured in nepers/meter [Np/m]. The ***neper*** is a dimensionless constant and defines the fraction of the attenuation the wave undergoes in 1 m. Attenuation of 1 neper/meter [Np/m] reduces the wave amplitude to $1/e$ as it propagates a distance of 1 m. Therefore, it is equivalent to 8.69 dB/m ($20 \log_{10} e = 8.69$), that is, 1 Np/m = 8.69 dB/m.

The imaginary part, β, only affects the phase of the wave and is called the ***phase constant***. The phase constant for lossless materials is identical to the wave number k as defined in **Eq. (12.31)**. However, we will use k as notation for wave number and use the phase constant β for all media, including lossless dielectrics.

A propagating wave in a lossy material is shown schematically in **Figure 12.13**. As the wave propagates in space, its amplitude is reduced exponentially. All aspects of propagation presented in the previous section remain the same except for replacing k by β and including the exponential decay in the amplitude.

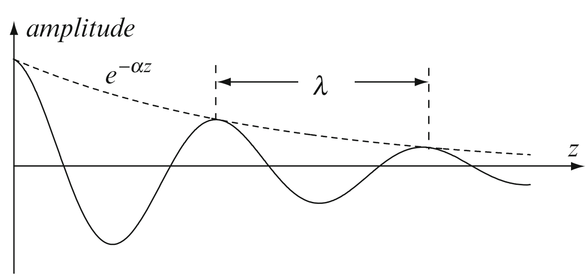

Figure 12.13 Propagation of a wave in a lossy material showing exponential attenuation

The attenuation and phase constants for a general, lossy material are found by separating the real and imaginary parts of γ in **Eq. (12.83)**. These are (see **Exercise 12.8**)

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right]} \tag{12.95}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.96}$$

The other parameters required for description of the wave in general lossy media are the phase velocity, wavelength, and intrinsic impedance. The phase velocity and wavelength are now

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}} \quad \left[\frac{\text{m}}{\text{s}}\right] \tag{12.97}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}} \quad [\text{m}] \tag{12.98}$$

Thus, both phase velocity and wavelength are smaller in lossy dielectrics, depending on conductivity. For lossless materials ($\sigma = 0$), **Eqs. (12.97)** and **(12.98)** reduce to those for lossless materials given in **Eqs. (12.28)** and **(12.30)**. To find the intrinsic impedance, we return to **Eqs. (12.35)** and **(12.36)**. The magnetic field intensity can be written from **Eq. (12.35)** as

$$\frac{\partial E_x^+}{\partial z} = \frac{\partial}{\partial z}\left(E_0^+ e^{-\gamma z}\right) = -\gamma E_x^+(z) \tag{12.99}$$

Substituting this in **Eq. (12.33)** gives

$$-\gamma E_x^+(z) = -j\omega\mu H_y \tag{12.100}$$

The intrinsic impedance is now written as

$$\eta = \frac{E_x^+(z)}{H_y^+(z)} = \frac{j\omega\mu}{\gamma} \quad [\Omega] \tag{12.101}$$

In the case considered here, the intrinsic impedance becomes

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \quad [\Omega] \tag{12.102}$$

The intrinsic impedance (also called the wave impedance) is now a complex number. It has both a resistive and a reactive part. In practical terms, this means that **E** and **H** are out of time phase in all but lossless materials and are out of phase in space for all materials (i.e., for plane waves they are perpendicular to each other).

Some important general observations are appropriate here:

(1) The phase velocity in lossy dielectrics is lower than in perfect dielectrics. This can be seen from **Eq. (12.97)** since β for lossy materials is larger than k for perfect dielectrics. Thus, the speed of propagation of electromagnetic waves is lower in lossy dielectrics (for the same ε and μ). The larger the losses, the lower the speed. For example, phase velocity in seawater is lower than distilled water because of its higher conductivity ($\sigma = 4$) (see also **Example 12.13**).

(2) The intrinsic impedance (wave impedance) in lossy dielectrics is complex, indicating a phase difference between the electric and magnetic field intensity in the same way as the phase difference between voltage and current in a circuit which contains reactive components. The magnitude of the intrinsic impedance is lower in conductive media. The higher the conductivity (losses), the lower the magnitude of the impedance.

(3) The electric and magnetic field intensity remain perpendicular to each other and to the direction of propagation regardless of losses. This is a property of the uniform plane waves we assumed.

(4) Attenuation of the wave in lossy media is exponential. This means that in materials with high conductivity, the attenuation is rapid. These materials will be called high-loss materials. Low-loss materials are materials with low conductivity.

Example 12.13 The electric field intensity of a plane electromagnetic wave is given as $\mathbf{E}(z) = \hat{\mathbf{x}}\, 8\cos(10^6 \pi t)$ [V/m] at a point $P(x = 0, y = 0, z = 0)$. The magnetic field intensity is in the positive y direction and the wave propagates in a material with properties $\varepsilon = \varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m], and $\sigma = 1.5 \times 10^{-5}$ S/m:

(a) Calculate the magnetic field intensity at a distance of 1 km from point P in the direction of propagation.
(b) How much faster does the wave propagate if $\sigma = 0$?

Solution: To evaluate the magnetic field intensity at point P, all we need is the intrinsic impedance in **Eq. (12.101)**. However, to evaluate the wave at a distance $z = 1{,}000$ m, we must also evaluate the attenuation and phase constants in **Eqs. (12.95)** and **(12.96)**.

(a) The intrinsic impedance of the material is

$$\eta = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\varepsilon_0}} = \sqrt{\frac{j10^6 \times \pi \times 4 \times \pi \times 10^{-7}}{1.5 \times 10^{-5} + j10^6 \times \pi \times 8.854 \times 10^{-12}}} = 342.69 + j86.5 \quad [\Omega]$$

The attenuation and propagation constants are

$$\alpha = \omega\sqrt{\frac{\mu_0\varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0}\right)^2} - 1\right]}$$

$$= 10^6 \times \pi\sqrt{\frac{4 \times \pi \times 10^{-7} \times 8.854 \times 10^{-12}}{2}\left[\sqrt{1 + \left(\frac{1.5 \times 10^{-5}}{10^6 \times \pi \times 8.854 \times 10^{-12}}\right)^2} - 1\right]} = 0.00273 \quad \left[\frac{\text{Np}}{\text{m}}\right]$$

$$\beta = \omega\sqrt{\frac{\mu_0\varepsilon_0}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_0}\right)^2} + 1\right]}$$

$$= 10^6 \times \pi\sqrt{\frac{4 \times \pi \times 10^{-7} \times 8.854 \times 10^{-12}}{2}\left[\sqrt{1 + \left(\frac{1.5 \times 10^{-5}}{10^6 \times \pi \times 8.854 \times 10^{-12}}\right)^2} + 1\right]} = 0.0108 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

In addition, the amplitude of the magnetic field intensity is

$$H(z) = \frac{E(z)}{\eta} \quad \left[\frac{\text{A}}{\text{m}}\right]$$

These are now combined together to write the magnetic field intensity. We will write it in the time domain. At point $P(0,0,0)$, the magnetic field intensity is

$$\mathbf{H}(z = 0) = \hat{\mathbf{y}}\frac{8}{\eta}\cos(10^6 \pi t) \quad \left[\frac{\text{A}}{\text{m}}\right]$$

At a distance z in the direction of propagation, the magnetic field intensity is

$$\mathbf{H}(z) = \hat{\mathbf{y}}\frac{8}{\eta}e^{-\alpha z}\cos\left(10^6\pi t - \beta z\right) \quad \left[\frac{\text{A}}{\text{m}}\right]$$

For the values given, this is

$$\mathbf{H}(z = 1000) = \hat{\mathbf{y}}\frac{8}{199.1\sqrt{2.777 + j1.5}}e^{-2.73}\cos\left(10^6\pi t - 10.8\right)$$
$$= \hat{\mathbf{y}}\,(0.0014 - j0.0004)e^{-2.73}\cos\left(10^6\pi t - 10.8\right) \quad \left[\frac{\text{A}}{\text{m}}\right]$$

The attenuation reduces the amplitude of the electric and magnetic fields to $\mathrm{e}^{-2.73} = 0.0652$ of their amplitude at $z = 0$ in 1 km (a factor of over 15). This attenuation is rather high, indicating a rather lossy dielectric.

(b) The phase velocity of the wave in free space (lossless) is $c = 3 \times 10^8$ m/s. In the lossy dielectric, the phase velocity is given in **Eq. (12.97)**:

$$v_p = \frac{\omega}{\beta} = \frac{10^6 \times \pi}{0.0108} = 2.91 \times 10^8 \quad \left[\frac{\text{m}}{\text{s}}\right]$$

Thus, the wave propagates about 3 % faster in free space than in the lossy dielectric.

Exercise 12.8 Show that **Eqs. (12.95)** and **(12.96)** are the real and imaginary parts of the propagation constant given in **Eq. (12.83)**.

Exercise 12.9 In the situation given in **Example 12.12**, assume that in addition to the properties given, there is also a small attenuation of 0.01 Np/m due to charged particles in air:

(a) Calculate the time-averaged power density in air at any location z.
(b) What are the stored magnetic and electric energy densities at the same location?

Answer

(a) $\mathcal{P}_{av} = \hat{\mathbf{z}}\dfrac{\eta_0 H_0^2 e^{-0.02z}}{2} \quad \left[\dfrac{\text{W}}{\text{m}^2}\right].$

(b) $w_m = \dfrac{\mu_0 H_0^2 e^{-0.02z}}{4}, \quad w_e = \dfrac{\varepsilon_0 \eta_0^2 H_0^2 e^{-0.02z}}{4} = \dfrac{\mu_0 H_0^2 e^{-0.02z}}{4} \quad \left[\dfrac{\text{J}}{\text{m}^3}\right].$

12.7.2 Propagation of Plane Waves in Low-Loss Dielectrics

We define low-loss materials as those materials in which the loss tangent is small: $\sigma/\omega\varepsilon \ll 1$ (or, equivalently, that the imaginary part of the complex permittivity in **Eq. (12.79)** is small compared to the real part). This relation also indicates that a material may be considered to be low loss at a given frequency range, whereas in another frequency range, this assumption may not hold. For example, conductivity of seawater is 4 S/m and its relative permittivity is 80 (at low frequencies). At 1 GHz $\sigma/\omega\varepsilon = 0.899$. At 100 GHz, $\sigma/\omega\varepsilon = 0.00899 \ll 1$. At 1 MHz, $\sigma/\omega\varepsilon = 899 \gg 1$. Thus, the classification of materials changes, depending on frequencies, but at 100 GHz, seawater is clearly low loss and at lower frequencies, it is a high-loss dielectric. In practice, the permittivity of the material also changes with frequency, changing the range in which a material may be considered to be a low-loss material.

All properties of the wave propagating in low-loss dielectrics remain the same as for any lossy material. But the above condition for low-loss materials simplifies some of these relations, allowing easier application and better understanding of behavior of waves propagating in these materials. The propagation constant now can be approximated using the binomial expansion (because $\sigma/\omega\varepsilon < 1$) as

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{\left(1-\frac{j\sigma}{\omega\varepsilon}\right)} \approx j\omega\sqrt{\mu\varepsilon}\left(1-\frac{j\sigma}{2\omega\varepsilon}+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2+\frac{j}{16}\left(\frac{\sigma}{\omega\varepsilon}\right)^3-\frac{5}{128}\left(\frac{\sigma}{\omega\varepsilon}\right)^4+\ldots\right) \tag{12.103}$$

Deciding, somewhat arbitrarily, to neglect all but the first three terms in the expansion, the attenuation constant is approximated by the second (real) term in **Eq. (12.103)**:

$$\boxed{\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}\eta_n \quad \left[\frac{\text{Np}}{\text{m}}\right]} \tag{12.104}$$

where η_n is the no-loss intrinsic impedance (i.e., the intrinsic impedance of a material with the same μ and ε but in which $\sigma = 0$), and the phase constant is

$$\boxed{\beta \approx \omega\sqrt{\mu\varepsilon}\left(1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right) \quad \left[\frac{\text{rad}}{\text{m}}\right]} \tag{12.105}$$

In very low-loss cases, the second term in **Eq. (12.105)** may also be neglected and the phase constant may often be approximated as

$$\beta \approx \omega\sqrt{\mu\varepsilon} \quad [\text{rad/m}] \tag{12.106}$$

The phase constant and, therefore, phase velocity and wavelength for low-loss dielectrics are essentially unchanged from those for the lossless dielectric because the second term in **Eq. (12.105)** is small, but the attenuation constant can be quite significant. Thus, the phase velocity and wavelength are

$$v_p \approx \frac{1}{\sqrt{\mu\varepsilon}\left(1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)} \quad \left[\frac{\text{m}}{\text{s}}\right], \quad \lambda = \frac{2\pi}{\beta} = \frac{v_p}{f} = \frac{1}{f\sqrt{\mu\varepsilon}\left(1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)} \quad [\text{m}] \tag{12.107}$$

for general low-loss materials and

$$v_p \approx \frac{1}{\sqrt{\mu\varepsilon}} \quad \left[\frac{\text{m}}{\text{s}}\right], \quad \lambda \approx \frac{2\pi}{\beta} = \frac{v_p}{f} \quad [\text{m}] \tag{12.108}$$

for very low-loss materials. The intrinsic impedance in low-loss dielectrics is still a complex number. Substituting the value of γ from **Eq. (12.103)** in **Eq. (12.101)**, and using the expansion again, η can be approximated as

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{\left(1-\frac{j\sigma}{\omega\varepsilon}\right)}} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1+\frac{j\sigma}{2\omega\varepsilon}\right) = \eta_n\left(1+\frac{j\sigma}{2\omega\varepsilon}\right) \quad [\Omega] \tag{12.109}$$

where η_n is the no-loss intrinsic impedance for the same material. The reactive part of the intrinsic impedance is quite small since $\sigma/\omega\varepsilon \ll 1$. Thus, for many practical applications, the intrinsic impedance of the lossless material may be used with little error.

Example 12.14 A satellite in geosynchronous orbit (36,000 km above the equator) is used for communication at 30 GHz. The atmosphere is 15 km thick. Assume free space above the atmosphere and plane wave propagation. Properties of the atmosphere are $\varepsilon = 1.05\varepsilon_0$ [F/m], $\mu = \mu_0$ [F/m], and $\sigma = 10^{-6}$ S/m:

(continued)

Example 12.14 (continued)

(a) Calculate the phase velocity in the atmosphere and in free space.
(b) Calculate the attenuation and phase constants in the atmosphere and in free space.
(c) Calculate the propagation constant in the atmosphere (air). Compare with the propagation constant in free space.
(d) Compare the intrinsic impedance in free space and in the atmosphere.
(e) If the minimum electric field intensity required for reception is 10 mV/m, what must be the minimum amplitude of the electric field intensity at the transmitter? Assume the satellite does not amplify the signal but only reflects it, and both transmitter and receiver are on Earth.

Solution: First, we check that the low-loss equations apply. If they do, the low-loss equations are used to calculate the phase velocity, attenuation, and phase constants. The attenuation constant is used to calculate the field after it propagates twice through the atmosphere (up and down). Only the atmosphere need be considered because there are no losses in free space:

(a) The low-loss condition is

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-6}}{2 \times \pi \times 3 \times 10^{10} \times 8.854 \times 1.05 \times 10^{-12}} = 5.7 \times 10^{-7} \ll 1$$

The low-loss approximation applies here. The phase velocity in free space is c = 3 × 10^8 m/s. In the atmosphere it is equal to $c/\sqrt{1.05} = 2.928 \times 10^8$ m/s. This is only about 2.4 % change in the phase velocity.

(b) The attenuation and phase constants are given in **Eqs. (12.104)** and **(12.106)**:

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \frac{\sigma\eta_0}{2\sqrt{\varepsilon_r}} = \frac{10^{-6} \times 377}{2 \times \sqrt{1.05}} = 1.84 \times 10^{-4} \quad \left[\frac{\text{Np}}{\text{m}}\right]$$

$$\beta \approx \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \omega\frac{\sqrt{\varepsilon}}{c} = 2 \times \pi \times 3 \times 10^{10}\frac{\sqrt{1.05}}{3 \times 10^8} = 643.83 \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

Note: The more accurate expression in **Eq. (12.105)** may be used but, because the loss is very low, an identical result is obtained. For most applications, unless $\sigma/\omega\varepsilon$ is close to 1, **Eq. (12.106)** should be used rather than **Eq. (12.105)**.
The attenuation constant in free space is zero. The phase constant in free space is

$$\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = 628.32 \quad \left[\frac{\text{rad}}{\text{m}}\right].$$

(c) The propagation constant in the atmosphere is $\gamma = \alpha + j\beta = 1.84 \times 10^{-4} + j643.83$, where α and β are those given in **(b)**. In free space, the propagation constant is $\gamma_0 = j\beta_0 = j628.32$.

(d) The intrinsic impedance in free space is 377 Ω. In the atmosphere, it is calculated using **Eq. (12.109)**, where $\eta_n = \eta_0/\sqrt{1.05}$

$$\eta \approx \eta_n\left(1 + \frac{j\sigma}{\omega\varepsilon}\right) = \frac{377}{\sqrt{1.05}}\left(1 + \frac{j10^{-6}}{2 \times 2 \times \pi \times 3 \times 10^{10} \times 1.05 \times 8.854 \times 10^{-12}}\right) = 367.9 + j1.05 \times 10^{-4} \quad [\Omega]$$

The intrinsic impedance has a small imaginary part. If we neglect this, the only change in the intrinsic impedance is due to change in permittivity.

(e) Because only the amplitude is required and this is only attenuated in the atmosphere, we write

$$E = 10 \times 10^{-3} = E_0 e^{-\alpha d} = E_0 e^{-1.84 \times 10^{-4} \times 30000} = 0.004E_0 \quad \to \quad E_0 = 2.5 \quad \left[\frac{\text{V}}{\text{m}}\right]$$

Note: This is an ideal example: there are losses in free space as well, but these are usually smaller than in the atmosphere. Because attenuation occurs mostly in the atmosphere, satellite communication requires relatively little power. If space were entirely lossless (which it is not because of existence of charged particles), communication with satellites would require about the same power levels as communication on Earth at distances of about 30 km (twice the assumed thickness of the atmosphere). In reality, the atmosphere is much thicker, but its density and losses diminish with altitude. Also, the transmission spreads over a relatively large area. Therefore, the required output from satellites is larger than that calculated here, but satellites with power outputs of between 100 W and 200 W are common.

12.7.3 Propagation of Plane Waves in Conductors

In highly conductive materials, the losses are high and we can assume that $\sigma \gg \omega\varepsilon$ or that the imaginary part of the complex permittivity is not negligible compared to the real part (i.e., conduction currents dominate). Under this condition, the complex propagation constant can be approximated from **Eq. (12.83)** as

$$\gamma \approx j\omega\sqrt{\mu\varepsilon}\sqrt{-\frac{j\sigma}{\omega\varepsilon}} = \sqrt{\frac{j\omega\mu\varepsilon\sigma}{\varepsilon}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} \tag{12.110}$$

by neglecting 1 compared to $j\sigma/\omega\varepsilon$ and using $\sqrt{\mathrm{j}} = (1+\mathrm{j})/\sqrt{2}$. From this, we get

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \quad \left[\frac{\mathrm{Np}}{\mathrm{m}}\right], \quad \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \quad \left[\frac{\mathrm{rad}}{\mathrm{m}}\right] \tag{12.111}$$

The attenuation and phase constants are equal in magnitude and are very large. The wave is attenuated rapidly to the point where propagation in conducting media can only exist within short distances. The propagating wave can now be written as

$$E_x(z) = E_0^+ e^{-z/\delta} e^{-jz/\delta} \quad [\mathrm{V/m}] \tag{12.112}$$

where the term

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}} = \frac{1}{\alpha} \quad [\mathrm{m}] \tag{12.113}$$

is known as the ***skin depth*** or ***depth of penetration*** of the wave. It is defined as that distance in which the amplitude of a plane wave is attenuated to $1/e$ of its original amplitude. The skin depth in conductors is small. In the microwave range, it can be of the order of a few microns (depending on material and frequency). Because waves at these high frequencies penetrate very little in conductors, it is quite common to use the perfect conductor approximation for conducting materials.

The phase velocity in good conductors is [from **Eqs. (12.97)** and **(12.111)**]

$$v = \frac{\omega}{\beta} = \omega\delta = \sqrt{\frac{2\omega}{\mu\sigma}} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{12.114}$$

and is obviously small compared to the phase velocity in dielectrics or free space, because δ is small.

The wavelength also changes drastically compared to free space or lossless dielectrics. It is very short and given by

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta \quad [\mathrm{m}] \tag{12.115}$$

The intrinsic impedance is [using **Eq. (12.102)**]

$$\eta \approx \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{(1+j)\sqrt{\omega\mu\sigma/2}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta} = (1+j)\frac{\omega\mu\delta}{2} \quad [\Omega] \tag{12.116}$$

where $j/(j+1) = (j+1)/2$ was used. The phase angle of the intrinsic impedance is, therefore, 45°. This is characteristic of good conductors for which the magnetic field intensity lags behind the electric field intensity by 45°. The intrinsic impedance of conductors can be very low and is much lower than the intrinsic impedance of free space. For example, the intrinsic impedance in copper at 1GHz is $(1 + j) \times 8.3 \times 10^{-3}\ \Omega$ compared to 377 Ω in free space.

Example 12.15 Application: Shielded Enclosures As an engineer you are asked to design a shielded room so that high-frequency waves cannot penetrate into the room. The shield is made of aluminum, in the form of a box. Assume that waves are plane waves and the lowest frequency at which the shielded room should satisfy the requirements is 1 MHz. The shield must reduce the amplitude of the electric fields inside the box by a factor of 10^6 compared to the amplitude outside. The conductivity of aluminum is 3.7×10^7 S/m. Assume that the electric field intensity at the outer surface of the conductor is the same as immediately below the surface and calculate the thickness required. This is not true in conductors because of reflection of waves at the surface. This subject will be treated in the next chapter, but for the purpose of this problem, we will assume no reflection at the surface of the conductor. The calculation here will give a "worst-case solution" since reflected waves will reduce the field in the conducting layer.

(a) Calculate the minimum thickness of the walls of the shielded room to satisfy the design criterion.
(b) Suppose you ran out of aluminum and needed to build the room with iron walls. If the conductivity of iron is 1×10^7 S/m and its relative permeability is 100, what is the thickness of the wall?

Solution: The amplitude of the electric field intensity is E_0 on one side of the conducting sheet and needs to be $10^{-6}E_0$ on the other side, as shown in **Figure 12.14**. The thickness necessary is calculated from the attenuation constant and the electric field intensities.

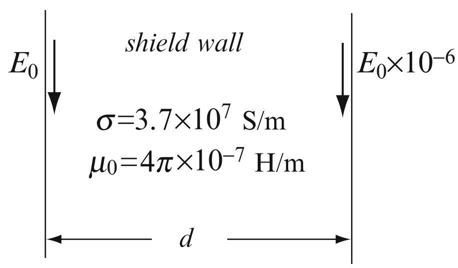

Figure 12.14 The relation between the electric field intensities at the two surfaces of a conducting shield wall

(a) The attenuation constant is given in **Eq. (12.111)**:

$$\alpha = \sqrt{\pi f \mu_0 \sigma} = \sqrt{\pi \times 10^6 \times 4 \times \pi \times 10^{-7} \times 10^7} = 1.21 \times 10^4 \quad [\text{Np/m}]$$

The electric field intensity at the inner and outer surfaces of the conducting sheet is related as

$$10^{-6}E_0 = E_0 e^{-\alpha d}$$

or taking the logarithm on both sides

$$-6\ln 10 = -\alpha d \quad \rightarrow \quad d = \frac{6\ln 10}{\alpha} = \frac{6 \times \ln 10}{1.21 \times 10^4} = 0.001142 \quad [\text{m}]$$

The aluminum box should be 1.142 mm thick.

(b) The conductivity of iron is lower but its permeability is higher. The attenuation constant in iron at the same frequency is

$$\alpha = \sqrt{\pi f 100 \mu_0 \sigma} = \sqrt{\pi \times 10^6 \times 400 \times \pi \times 10^{-7} \times 10^7} = 6.28 \times 10^4 \quad [\text{Np/m}]$$

and the thickness d is

$$d = \frac{6\ln 10}{\alpha} = \frac{6 \times \ln 10}{6.28 \times 10^4} = 0.00022 \quad [\mathrm{m}]$$

Iron is a better shielding material because the iron shield is over 5 times thinner, and, even though iron is about 2.5 times heavier than aluminum, the total weight is lower. Higher-permeability materials may be used to obtain an even more effective shield. On the other hand, high conductivity, nonmagnetic materials like copper and aluminum are commonly used for shields at high frequencies because they are easier to work with and have other properties that make them attractive for design. This calculation shows why your radio will not work inside a reinforced concrete garage or, say, underwater.

Example 12.16 Application: Communication with Submarines The most severe restriction to communication with submerged submarines is the high loss exhibited by seawater. The following example shows this difficulty.

Suppose we wish to communicate with submarines using a conventional communication system at 1 MHz (in the AM radio range). Properties of seawater are $\varepsilon = 81\varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m], and $\sigma = 4$ S/m. Assume that a magnetic field intensity of 10,000 A/m can be generated at the surface of the ocean and that the receiver in the submarine can receive magnetic fields as low as 1 μA/m:

(a) Calculate the maximum range for this communication system.
(b) Suppose the frequency is lowered to 100 Hz. What is the range now?
(c) Suppose the antenna on the submarine must be one half wavelength in length. Calculate the required antenna lengths in **(a)** and in **(b)**.

Solution: First, we must check if seawater should be treated as a conductor or as a lossy dielectric. Then we need to calculate the attenuation constant and the range for communication as in **Example 12.15**. The same applies to the lower frequency in **(b)**, but we need to check again if seawater can be treated as a conductor at the lower frequency:

(a) To see if seawater is a conductor or not at 1 MHz, we write

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2 \times \pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888 \gg 1$$

This is a high-loss medium and the attenuation constant is

$$\alpha = \sqrt{\pi f \mu_0 \sigma} = \sqrt{\pi \times 10^6 \times 4 \times \pi \times 10^{-7} \times 4} = 3.974 \quad [\mathrm{Np/m}]$$

The maximum range is calculated from the following relation (see **Example 12.15**):

$$1 \times 10^{-6} = H_0 e^{-\alpha d} = 10000 e^{-\alpha d} \quad \rightarrow \quad 1 \times 10^{-10} = e^{-\alpha d} \quad \rightarrow \quad d = \frac{10\ln 10}{3.974} = 5.794 \quad [\mathrm{m}]$$

The range for communication is less than 6 m. This perhaps is not surprising since the skin depth in seawater at 1 MHz is only about 0.25 m.

(b) At 100 Hz, we have

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2 \times \pi \times 100 \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^6 \gg 1$$

and this is certainly a conductor. The attenuation constant is

$$\alpha = \sqrt{\pi f \mu_0 \sigma} = \sqrt{\pi \times 100 \times 4 \times \pi \times 10^{-7} \times 4} = 0.03974 \quad [\mathrm{Np/m}]$$

Using the formula for distance in **(a)**, we get

$$d = \frac{10 \ln 10}{0.03974} = 579.4 \quad [\text{m}]$$

This is not a very long range but is feasible for communication underwater.

Note: Seawater is unique in that it is a high-loss dielectric. Its permittivity goes down with frequency, but its attenuation is very high and increases with frequency as this example shows. For this reason, communication in seawater is very difficult, and if done, it must be done at very low frequencies.

(c) The wavelength is calculated as

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\alpha} \quad [\text{m}]$$

The wavelength (underwater) at 100 Hz is 158 m and at 1 MHz it is 1.58 m. A half-wavelength antenna at 100 Hz must be 79 m long and at 1 MHz it must be 0.79 m. Note that the wavelength underwater is significantly shorter than in free space.

Low-frequency trailing antennas for submarines are used for communication at very low frequencies (usually below 150 Hz). These antennas may be hundreds of meters long.

Example 12.17 Application: Current Distribution and AC Resistance of Conductors One important consequence of the skin depth is the fact that currents in conductors decay exponentially from the surface inward. Thus, in AC systems, the current-carrying capacity is reduced since more of the current flows on the surface, whereas the current density allowable is fixed. This also means that the AC resistance of a conductor is larger than its DC resistance.

A conductor with a 4 mm diameter carries a total current $I = 100$ A. The conductor is made of copper with a conductivity of 5.7×10^7 S/m. Calculate:

(a) DC resistance per meter length and current density in the wire.

(b) AC resistance per meter length and maximum current in the wire. Use a frequency of 60 Hz and assume exponential decay of the current density from the surface to the center of the wire with maximum current density allowable being the same as in the DC case.

Solution: The electric field intensity in the conductor decays exponentially from its value on the surface. Since $\mathbf{J} = \sigma\mathbf{E}$, the current density $\mathbf{J}$ also decays exponentially. Thus, to calculate the DC and AC resistances, we assume in each case a known current density at the surface of the conductor. In the DC case, this current density is uniform throughout the conductor's cross section. In the AC case, it is largest at the surface. Therefore, for a given current density, the total current in the DC case is larger and its DC resistance is lower:

(a) The DC resistance of the conductor was calculated in **Chapter 7 [Eq. (7.18)]** as

$$R_{dc} = \frac{L}{\sigma S} = \frac{1}{5.7 \times 10^7 \times \pi \times 0.002^2} = 0.001396 \quad \left[\frac{\text{W}}{\text{m}}\right]$$

The current density in the conductor is the total current divided by its cross-sectional area:

$$J = \frac{I}{S} = \frac{100}{\pi \times 0.002^2} = 7.958 \times 10^6 \quad \left[\frac{\text{A}}{\text{m}^2}\right].$$

(b) To calculate the AC resistance, we assume this current density at the surface. Then, we calculate the current density everywhere in the conductor and integrate the current density to obtain the total current. The same voltage that produced the current in **(a)** must produce the current in **(b)**. Thus, the resistance is calculated from the total current and voltage. The attenuation constant is

$$\alpha = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 60 \times 4 \times \pi \times 10^{-7} \times 5.7 \times 10^7} = 116.2 \quad \left[\frac{\text{Np}}{\text{m}}\right]$$

The skin depth in the conductor is

$$\delta = \frac{1}{\alpha} = 0.0086 \quad [\text{m}]$$

The current density in the conductor can now be written as

$$J(r) = J_0 e^{-\alpha(a-r)}$$

where J_0 is the current density at the surface, a is the radius of the conductor, and r is the distance from the center of the conductor at which the current density is calculated. The total current in the conductor is now calculated using **Figure 12.15**. The current in a ring of radius r and thickness dr is

$$dI_{ac} = 2\pi r dr J(r) = 2\pi r J_0 e^{-\alpha(a-r)} dr$$

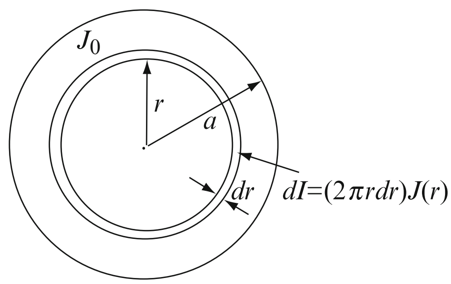

Figure 12.15 Calculation of AC current in a conductor

Integrating over the radius of the conductor

$$I_{ac} = 2\pi J_0 e^{-\alpha a} \int_{r=0}^{r=a} r e^{\alpha r} dr = 2\pi J_0 e^{-\alpha a} \frac{e^{\alpha r}}{\alpha^2}(\alpha r - 1)\Big|_0^a$$

this gives

$$I_{ac} = 2\pi J_0 e^{-\alpha a}\left[\frac{e^{\alpha a}}{\alpha^2}(\alpha a - 1) + \frac{1}{\alpha^2}\right]$$
$$= 2 \times \pi \times 7.958 \times 10^6 e^{-116.2\times 0.002}\left[\frac{e^{116.2\times 0.002}}{116.2^2}(116.2 \times 0.002 - 1) + \frac{1}{116.2^2}\right] = 92.7 \quad [\text{A}]$$

The voltage that produced the current in **(a)** is $V = I_{dc}R_{dc}$. If we connect an AC voltage equal in magnitude to the DC voltage in **(a)** to a 1 m length of the conductor, the AC resistance is

$$R_{ac} = \frac{V}{I_{ac}} = \frac{I_{dc}R_{dc}}{I_{ac}} = \frac{0.00139 \times 100}{92.7} = 0.0015 \quad \left[\frac{\Omega}{\text{m}}\right]$$

The AC resistance is about 8 % higher than the DC resistance, causing more losses on the line.

This example shows a very important aspect of power transmission; DC systems are more efficient in transferring power. For the same allowable current density in a conductor, they transfer more power than AC systems. However, because of questions of transformation of voltages, AC power is usually used for transmission. When power must be transmitted over long distances, high-voltage direct current (HVDC) systems are more economical in spite of the need for converters and inverters on the ends of the power transmission line. High-voltage direct current systems are in use in many countries, especially for transmission to large cities and industrial centers from remote power stations. Voltages in excess of 1 million volts are used for this purpose.

12.7.4 The Speed of Propagation of Waves and Dispersion

In **Section 12.3.5**, we defined the phase velocity and wave number of a plane wave propagating in free space. These definitions were extended to perfect dielectrics, lossy dielectrics, and conductors in **Sections 12.7.1** through **12.7.3**. However, there are some difficulties with the definition of phase velocity which we have not addressed and which will be discussed here. First, the phase velocity, defined as $v_p = 1/\sqrt{\mu\varepsilon}$ [see **Eq. (12.29)**], cannot be used in general. For example, we had to modify v_p in lossy dielectrics and conductors as shown in **Eqs. (12.97)** and **(12.107)**, respectively. Clearly, the definition in **Eq. (12.29)** is only valid in free space and, by extension, in lossless dielectrics.

Similarly, the wave number $k = \omega\sqrt{\mu\varepsilon}$ in **Eq. (12.23)** or the phase constant β in **Eq. (12.96)** is frequency dependent; that is, for one frequency, the phase changes differently than for another. Thus, if the wave consists of more than a single frequency (the wave is not monochromatic), there will be a distortion of the wave as it propagates. The definition of phase velocity as used up to now, of course, only applied to monochromatic waves. Can we still use the idea of a phase velocity for nonmonochromatic waves? The answer is clearly no, except for propagation in lossless unbounded media. To characterize the speed of propagation of a nonmonochromatic wave, we will introduce here the idea of group velocity. Finally, in this regard, we might ask the following question: Are the phase velocity or group velocity also the speed at which energy propagates? Again, the answer, in general, is no, and we will have to define a new velocity: the velocity of energy transport to answer this question.

The meaning of phase velocity is usually taken as the speed with which the phase of a wave propagates in space. It was alluded to in **Section 12.3.5** that this is not the speed of any real quantity propagating and therefore can be, and often is, larger than the speed of light.

Fortunately, there are many cases in which the many velocities of electromagnetic waves (or light) yield the same results and there is no need to worry about different velocities. In particular, in monochromatic plane waves, propagating in free space in an unbounded domain, the phase velocity, group velocity, and velocity of energy transport are the same. Similarly, in lossless dielectrics as well as in very low-loss dielectrics, either the phase velocity is independent of frequency or may be approximated as frequency independent, simplifying analysis.

In addition to phase velocity, group velocity, and the velocity of energy transport, there are other definitions of wave velocity in electromagnetics, each with its own assumptions and uses.[4]

12.7.4.1 Group Velocity

Group velocity is the velocity of a wave packet consisting of a narrow range or band of frequencies. An example akin to this is a frequency-modulated (FM) wave as used in FM radio transmission. In this type of wave, a carrier wave at an angular frequency ω_0 is modulated by another wave of angular frequency $\Delta\omega \ll \omega_0$. The angular frequency of the wave will vary between $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$. Clearly, we cannot now talk about the phase velocity of the wave because phase velocity is only defined for a single frequency.

To define the group velocity of a packet of waves, we will consider here the case of amplitude modulation (AM). Consider two waves with the same amplitude and propagating in the same direction, but the two waves are at slightly different frequencies. One wave is at an angular frequency $\omega_1 = \omega_0 - \Delta\omega$ and the other at $\omega_2 = \omega_0 + \Delta\omega$. The amplitudes of the waves are E and the waves propagate in the z direction in a lossless medium. The phase constants of each of the waves are written from the definition $\beta = \omega\sqrt{\mu\varepsilon}$ [rad/m]. Therefore, $\beta_1 = \beta_0 - \Delta\beta$ and $\beta_2 = \beta_0 + \Delta\beta$. With these, the waves are

$$E_1(z,t) = E\cos(\omega_1 t - \beta_1 z) = E\cos\big((\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z\big) \quad [\mathrm{V/m}] \tag{12.117}$$

$$E_2(z,t) = E\cos(\omega_2 t - \beta_2 z) = E\cos\big((\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z\big) \quad [\mathrm{V/m}] \tag{12.118}$$

[4] R. L. Smith, "The Velocities of Light," American Journal of Physics, Vol. 38, No. 8, Aug. 1970, pp. 978–983.

The sum of these two waves gives the total wave:

$$\begin{aligned} E(z,t) &= E_1(z,t) + E_2(z,t) \\ &= E\cos((\omega_0+\Delta\omega)t - (\beta_0+\Delta\beta)z) + E\cos((\omega_0-\Delta\omega)t - (\beta_0-\Delta\beta)z) \\ &= E\cos(\omega_0+\Delta\omega)t\cos(\beta_0+\Delta\beta)z + E\sin(\omega_0+\Delta\omega)t\sin(\beta_0+\Delta\beta)z \\ &\quad + E\cos(\omega_0-\Delta\omega)t\cos(\beta_0-\Delta\beta)z + E\sin(\omega_0-\Delta\omega)t\sin(\beta_0-\Delta\beta)z \\ &= [2E\cos(\Delta\omega t - \Delta\beta z)]\cos(\omega_0 t - \beta_0 z) \quad [\mathrm{V/m}] \end{aligned} \tag{12.119}$$

This is a wave with amplitude equal to the sum of the amplitudes of the individual waves and a fundamental or carrier frequency ω_0. The amplitude of the wave varies cosinusoidally with frequency $\Delta\omega$ as can be seen in **Figure 12.16**. The carrier travels at a velocity v_p, which is calculated **[**see **Eq. (12.28)]** as follows.

By assuming a constant point on the carrier, the phase velocity of the single frequency carrier is

$$\omega_0 t - \beta_0 z = const. \quad \rightarrow \quad \frac{d(\omega_0 t - \beta_0 z)}{dt} = \omega_0 - \beta_0\frac{dz}{dt} = 0 \quad \rightarrow \quad v_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{12.120}$$

This much we have seen for a monochromatic plane wave.

The modulation, or the envelope, also travels but at a different velocity. Performing the same operation for the modulation, we write

$$\Delta\omega t - \Delta\beta z = \mathrm{const.} \quad \rightarrow \quad \frac{d(\Delta\omega t - \beta z)}{dt} = \Delta\omega - \Delta\beta\frac{dz}{dt} = 0 \quad \rightarrow \quad v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{12.121}$$

In the limit, as $\Delta\beta$ tends to zero, we write

$$\boxed{v_g = \lim_{\Delta\beta\to 0}\frac{1}{\Delta\beta/\Delta\omega} = \frac{1}{d\beta/d\omega} \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]} \tag{12.122}$$

This velocity is called the ***group velocity*** or the velocity of the wave packet with a narrow frequency width ($\Delta\omega \ll \omega_0$). The latter is actually an informative name since the modulation, or the group, is traveling at this velocity, which can be very different than the phase velocity. The definition given here does not apply to wide-band signals.

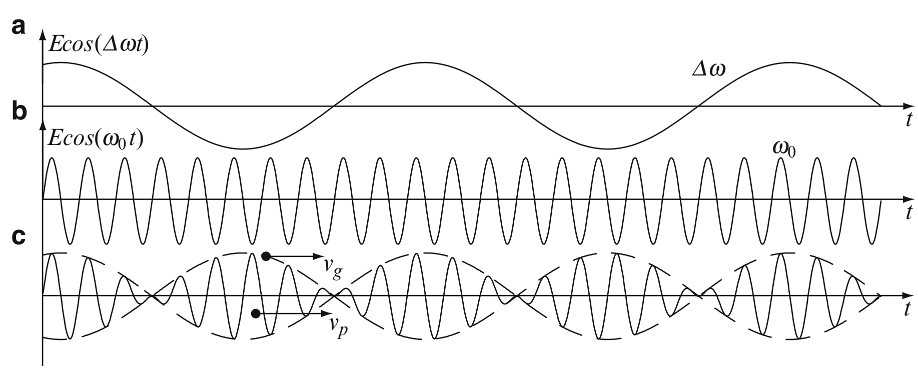

Figure 12.16 Amplitude modulation. (**a**) The modulating signal. (**b**) The high-frequency carrier. (**c**) The amplitude-modulated carrier. Signals shown at $z = 0$

12.7.4.2 Velocity of Energy Transport

The velocity of energy transport is based on the fact that when calculating the Poynting vector of a wave, we, in fact, calculate the rate of change of energy in a given volume. The energy density in the volume is given in [J/m^3], whereas the Poynting vector is given in [W/m^2]. The ratio between the Poynting vector and the energy density is the velocity of energy transport. In a lossless medium, the velocity of energy transport is defined as the ratio between the time-averaged propagated power density (time-averaged Poynting vector $\mathcal{P}_{av}$) and the time-averaged stored energy density:

$$\mathbf{v}_e = \mathcal{P}_{av}/w_{av} \quad [\mathrm{m/s}] \tag{12.123}$$

This definition of velocity is convenient because it gives the velocity at which energy is transported and is always lower or equal to the speed of light in the medium in which the wave propagates. As such, it is in agreement with our idea of all speeds being limited by the speed of light. In addition, the direction of propagation of energy is also immediately available. This definition will become handy in cases where the phase velocity becomes larger than the speed of light, a situation often encountered in the propagation of waves in the presence of conducting bodies (but not inside the conductors). Because the velocity of energy transport is defined for lossless media, it is always equal to the phase velocity in lossless, unbounded space. It is, however, different when waves propagate in bounded space, as we shall see in the next chapter.

12.7.4.3 Dispersion

Now that we have the group and phase velocities, it is relatively easy to understand what dispersion means. To see this, consider again the two waves in **Eqs. (12.117)** and **(12.118)**, but suppose that each propagates a distance z_0, separately, in a lossy dielectric. The phase velocity of each wave is given in **Eq. (12.97)** and is different for each wave. The two waves arrive at their target at different times and with different phase angles. If the two waves carry information, then this information will arrive distorted. A simple example is transmission of music from a radio station to your radio receiver. Each frequency in the signal will propagate at different speeds and arrive with different phases. The signal you will hear will have components which are delayed and shifted in phase. If the phase velocity is frequency dependent, we say that the signal disperses and, therefore, that the medium through which the wave propagates is dispersive. We call any material in which the phase velocity depends on frequency a ***dispersive medium***. Fortunately, not all materials are dispersive. We saw that in free space, the phase constant is linearly dependent on frequency ($\beta = \omega\sqrt{\mu\varepsilon}$). Therefore, the phase velocity $v_p = \omega/\beta = 1/\sqrt{\mu\varepsilon}$ is independent of frequency. Free-space and perfect dielectrics are nondispersive media. In nondispersive media, all waves propagate at the same speed and, therefore, the group velocity and the phase velocity are equal: $v_g = v_p$. In other materials, the permittivity and, sometimes, the permeability are frequency dependent; therefore, these materials are dispersive.

In most materials, the phase velocity decreases as frequency increases. This dispersion is called ***normal dispersion***. In some other materials, the phase velocity increases with frequency. This is called ***anomalous dispersion***.

A dispersion relation is the relation between β and ω shown in **Figure 12.17**. **Figure 12.17a** shows a number of nondispersive materials. The relation between β and ω is linear. Therefore, when taking the derivative in **Eq. (12.122)**, the result is a straight line whose slope is the phase velocity in the medium. **Figure 12.17b** shows a nonlinear dispersion relation. The group velocity is the tangent to this line at any point on the line, indicating that the group velocity is frequency dependent. The line asymptotically tangential to the curve is the lossless phase velocity (since, as frequency approaches infinity, $\sigma/\omega\varepsilon$ approaches zero and the material becomes lossless).

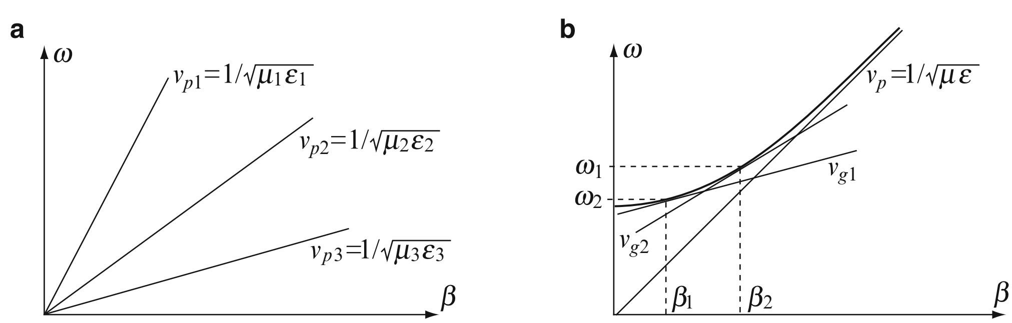

Figure 12.17 Dispersive and nondispersive media. (**a**) A nondispersive medium has a linear relation between frequency and phase constant. (**b**)A dispersive medium has a nonlinear relation between frequency and phase constant

Example 12.18 Application: Dispersion in the Atmosphere A TV station operates with a carrier signal of 96 MHz. The video signal, which is modulated on the carrier signal, is 6 MHz in width, making the frequency of the wave vary between 93 and 99 MHz. The waves generated by the station may be viewed as plane waves at large distances from the station. Assume the station transmits in a lossy atmosphere with permeability μ_0, permittivity $1.05\varepsilon_0$, and conductivity 10^{-3} S/m:

(a) Calculate the phase and group velocities of the wave.
(b) Show that if we can assume the medium to be a very low-loss material, phase velocity and group velocity are essentially the same.

Solution: Before doing anything else, we must decide which approximations, if any, may be used for calculations by evaluating $\sigma/\omega\varepsilon$. Based on this, we choose the appropriate formulas for phase velocity, phase constant, etc. The group velocity is then calculated from the phase constant using **Eq. (12.122)**.

In this case,

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-3}}{2 \times \pi \times 96 \times 10^6 \times 1.05 \times 8.854 \times 10^{-12}} = 0.178 < 1.0$$

However, it would be inappropriate to assume that this is a very low-loss dielectric and, therefore, we must use **Eq. (12.105)** for the phase constant and **Eq. (12.107)** for phase velocity.

(a) The phase velocity is frequency dependent, as can be seen from **Eq. (12.107)**. Thus, we calculate the minimum (at the lowest frequency transmitted by the station) and maximum phase velocities (at the highest frequency transmitted):
At 93 MHz:

$$v_{p\min} \approx \frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega_{\min}\varepsilon}\right)^2\right)} = \frac{c}{\sqrt{\varepsilon_r}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega_{\min}\varepsilon}\right)^2\right)}$$

$$= \frac{3 \times 10^8}{\sqrt{1.05}\left(1 + \frac{1}{8}\left(\frac{10^{-3}}{2 \times \pi \times 93 \times 10^6 \times 1.05 \times 8.854 \times 10^{-12}}\right)^2\right)} = 2.9154 \times 10^8 \quad \left[\frac{\text{m}}{\text{s}}\right]$$

At 99 MHz:

$$v_{p\,max} \approx \frac{c}{\sqrt{\varepsilon_r}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega_{max}\varepsilon}\right)^2\right)}$$

$$= \frac{3 \times 10^8}{\sqrt{1.05}\left(1 + \frac{1}{8}\left(\frac{10^{-3}}{2 \times \pi \times 99 \times 10^6 \times 1.05 \times 8.854 \times 10^{-12}}\right)^2\right)} = 2.9168 \times 10^8 \quad \left[\frac{\text{m}}{\text{s}}\right]$$

Although the velocities only differ by about 0.06 %, this can cause distortions, especially if transmission is over long distances.

To calculate the group velocity, we use **Eq. (12.122)** with the phase constant given in **Eq. (12.105)**. From this, we evaluate

$$\frac{\partial\beta}{\partial\omega} = \frac{\partial}{\partial\omega}\left[\omega\sqrt{\mu\varepsilon}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)\right] = \sqrt{\mu\varepsilon}\left(1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)$$

The group velocity is then given from **Eq. (12.122)** as

$$v_{\mathrm{g}} = \frac{1}{\partial\beta/\partial\omega} = \frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}\left(1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)} = \frac{c}{\sqrt{\varepsilon_{\mathrm{r}}}\left(1 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)}$$

$$= \frac{3\times10^8}{\sqrt{1.05}\left(1-\frac{1}{8}\left(\frac{10^{-3}}{2\times\pi\times96\times10^6\times1.05\times8.854\times10^{-12}}\right)^2\right)} = 2.939384\times10^8 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

Note that the group velocity is calculated at the center (carrier) frequency and is different than the phase velocity at that same frequency which is 2.9161×10^8 m/s. However, the differences are small and for this reason, we usually cannot see distortions in TV transmissions due to dispersion.

(b) In a very low-loss material, $\sigma/\omega\varepsilon$ can be neglected and the phase velocity becomes that of the lossless medium. Under this condition,

$$v_p = v_g = \frac{1}{\sqrt{\mu\varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{1.05}} = 2.9277\times10^8 \quad \left[\frac{\mathrm{m}}{\mathrm{s}}\right].$$

12.8 Polarization of Plane Waves

The electric (or magnetic) field intensity of a uniform plane wave has a direction in space. This direction may either be constant or may change as the wave propagates. The polarization of a plane wave is

"*the figure traced by the tip of the electric field vector as a function of time, at a fixed point in space*."

Polarization is defined for the electric field vector only, since the magnetic field is always obtainable from the electric field by use of Maxwell's equations [(**Eq. (12.32)**] and there is no need to define its polarization separately. The reason why polarization of plane waves is important is because the propagation properties of the wave are affected by the polarization. A simple example is the direction of an antenna. If the electric field is, say, horizontally directed in space (polarized in this direction), then a receiving antenna must also be directed horizontally if the electric field intensity (and therefore the current) in the antenna is to be maximized. Polarization of plane waves can be intentional or a result of propagation through materials.

We will treat linear polarization first as the simplest polarization and then extend linear polarization to include the important cases of circular and elliptical polarization.

The polarization of a plane wave is determined as follows:

(a) Write the electric field intensity in the time domain.

(b) An observation point in space is chosen so that the wave propagates straight toward the observer.

(c) The direction of the electric field intensity **E** is followed as time changes. The tip of the vector traces some pattern in the plane perpendicular to the direction of propagation. This trace is the polarization of the wave. If the figure traced by the tip of the electric field intensity is a straight line, we call this ***linear polarization***. In general, the trace is an ellipse and the polarization is said to be ***elliptical polarization***. A special case of the elliptical polarization is ***circular polarization***.

(d) In addition, the vector may seem to rotate clockwise or counterclockwise as time changes. If the wave rotates in a clockwise direction, the wave is said to be ***left-hand polarized***. If the wave rotates in a counterclockwise direction, it is said to be ***right-hand polarized***. A wave may be, for example, ***left-hand elliptically polarized*** (or, in short, ***left elliptically polarized***) or ***right-hand elliptically polarized*** (in short, ***right elliptically polarized***). The sense of rotation of the wave is determined by the simple use of the right-hand rule: If the fingers of the right-hand curl in the direction of rotation of the electric field, the thumb must show in the direction of propagation for a right-hand sense and in the direction opposite the direction of propagation for a left-hand sense.

These definitions and their physical meaning are described next.

12.8.1 Linear Polarization

The electric field intensity

$$\mathbf{E} = \hat{\mathbf{y}} E_y(z) = \hat{\mathbf{y}} E_y e^{-\gamma z} \quad [\text{V/m}] \tag{12.124}$$

of a wave propagating in the z direction and directed in the y direction is linearly polarized in the y direction. The electric field intensity varies in the y direction (its amplitude may be constant or decaying, but, as it propagates, the phase changes). This linearly polarized wave is shown in **Figure 12.18a** as viewed by an observer, placed on the z axis looking onto the xy plane where the electric field is shown as it changes with time. To see how this occurs, we write the electric field intensity in **Eq. (12.124)** in the time domain:

$$\mathbf{E}(z,t) = \text{Re}\left\{\hat{\mathbf{y}} E_y e^{-\gamma z} e^{j\omega t}\right\} = \text{Re}\left\{\hat{\mathbf{y}} E_y e^{-\alpha z - j\beta z} e^{j\omega t}\right\} = \hat{\mathbf{y}} E_y e^{-\alpha z}\cos(\omega t - \beta z) \quad [\text{V/m}] \tag{12.125}$$

For any constant value of z, the direction of the vector remains in the y direction as the amplitude changes between a negative value $-E_y e^{-\alpha z}$ to a positive value $E_y e^{-\alpha z}$.

The electric field intensity

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x e^{-\alpha z}\cos(\omega t - \beta z) + \hat{\mathbf{y}} E_y e^{-\alpha z}\cos(\omega t - \beta z) \quad [\text{V/m}] \tag{12.126}$$

has two components: one in the x direction, one in the y direction. **Figure 12.18b** shows that each component describes a line on the corresponding axis while its amplitude is attenuated. The resultant wave is in a direction that depends on the amplitudes E_x and E_y. As an example, if E_x and E_y are equal, the electric field is polarized at 45° to the x axis. In general, the superposition of two fields, or two components of the same field, each with linear polarization, produces a linearly polarized wave at an angle $\tan^{-1}(E_y/E_x)$, if the two electric fields are in time phase (or $\tan^{-1}(E_y/E_x)$, if the two are 180° out of time phase).

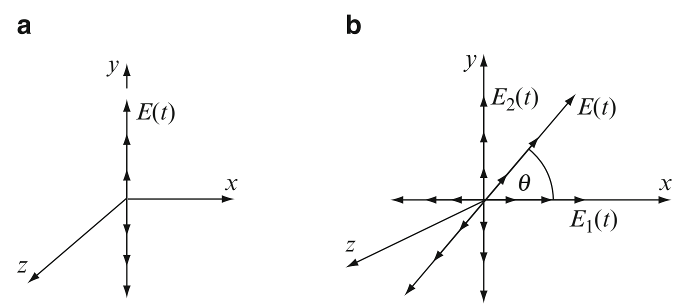

Figure 12.18 (**a**) Time variation of a wave, linearly polarized in the y direction traveling in the z direction. (**b**) General linear polarization obtained by superposition of two linearly polarized waves which are in time phase

12.8.2 Elliptical and Circular Polarization

Suppose an electromagnetic wave has an electric field intensity with two components in space, say one in the x direction and one in the y direction. The two components are out of time phase with the y component leading the x component by an angle φ. The electric field intensity in the time domain is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_1\cos(\omega t - \beta z) + \hat{\mathbf{y}} E_2\cos(\omega t - \beta z + \varphi) \quad [\text{V/m}] \tag{12.127}$$

This is simply the superposition of the two components of the electric field. To define the polarization and sense of rotation of the wave, we use the above four steps:

Step a: Write the equation in the time domain **[Eq. (12.127)]**.

Step b: Fix a value for z. In most cases, $z = 0$ is most convenient, for which we have

$$\mathbf{E}(z=0,t) = \hat{\mathbf{x}} E_1\cos(\omega t) + \hat{\mathbf{y}} E_2\cos(\omega t + \varphi) \quad [\text{V/m}] \tag{12.128}$$

Now, the values defining the behavior of the equation are the amplitudes E_1 and E_2 and the phase angle φ. Although these may have any values in general, we distinguish two important values for E_1 and E_2, namely, $E_1 \neq E_2$ and $E_1 = E_2$, and two values for φ, $\varphi = -\pi/2$ and $\varphi = +\pi/2$. This defines four distinct cases as follows:

Case 1: $E_1 \neq E_2$ and $\varphi = -\pi/2$. The electric field intensity now is

$$\mathbf{E}(z=0,t) = \hat{\mathbf{x}}E_1\cos(\omega t) + \hat{\mathbf{y}}E_2\cos\left(\omega t - \frac{\pi}{2}\right) = \hat{\mathbf{x}}E_1\cos(\omega t) + \hat{\mathbf{y}}E_2\sin(\omega t) \quad [\text{V/m}] \tag{12.129}$$

Two relations may be extracted from this equation. If we denote $E_x = E_1\cos\omega t$ and $E_y = E_2\sin\omega t$, we can write

$$\cos(\omega t) = \frac{E_x(0,t)}{E_1}, \quad \sin(\omega t) = \frac{E_x(0,t)}{E_2} \tag{12.130}$$

and

$$\cos(\omega t) + \sin^2(\omega t) = 1 \quad \rightarrow \quad \frac{E_x^2(0,t)}{E_1^2} + \frac{E_y^2(0,t)}{E_2^2} = 1 \tag{12.131}$$

Step c: **Equation** (12.131) is the equation of an ellipse; that is, as time changes, the tip of the electric field intensity **E** (whose components are $E_x(0,t)$ and $E_y(0,t)$) describes an ellipse on the $x-y$ plane. The wave is therefore elliptically polarized, as shown in **Figure** 12.19a.

Step d: To find the sense of rotation of the field vector, we use the following simple method. Using the time domain field expression [**Eq.** (12.129) in this case], two values of ωt are substituted and the vector **E** evaluated. From these, it is possible to determine the rotation. For example, we may choose convenient values $\omega t = 0$ and $\omega t = \pi/2$. These give

$$\mathbf{E}(z=0,\omega t=0) = \hat{\mathbf{x}}E_1, \quad \mathbf{E}(z=0,\omega t=\pi/2) = \hat{\mathbf{y}}E_2 \quad [\text{V/m}] \tag{12.132}$$

As ωt changed from 0 to $\pi/2$, the electric field vector moved counterclockwise from being in the positive x direction to the positive y direction (see **Figure** 12.19b). Using the right-hand rule, the wave is found to be a right elliptically polarized wave. This method is quite simple but it is important to choose values of ωt that are convenient for calculation and that give unambiguous answers. If, for example, we found that at the second value of ωt the vector was in the negative x direction, it would have been impossible to determine if the rotation is clockwise or counterclockwise.

Case 2: $E_1 \neq E_2$ and $\varphi = \pi/2$. Again, we follow the above five steps. The electric field intensity in the time domain (**step a**) is given in **Eq.** (12.128). Setting $z = 0$ (**step b**) and $\varphi = +\pi/2$ gives the electric field intensity as

$$\mathbf{E}(z=0,t) = \hat{\mathbf{x}}E_1\cos(\omega t) + \hat{\mathbf{y}}E_2\cos\left(\omega t + \frac{\pi}{2}\right) = \hat{\mathbf{x}}E_1\cos(\omega t) - \hat{\mathbf{y}}E_2\sin(\omega t) \quad [\text{V/m}] \tag{12.133}$$

Equation (12.131) remains the same and, therefore, the wave is still elliptically polarized. To find the sense of rotation, we again choose two values of ωt; $\omega t = 0$ and $\omega t = \pi/2$, and substitute in the expression for **E** in **Eq.** (12.133). These give

$$\mathbf{E}(z=0,\omega t=0) = \hat{\mathbf{x}}E_1, \quad \mathbf{E}(z=0,\omega t=\pi/2) = -\hat{\mathbf{y}}E_2 \quad [\text{V/m}] \tag{12.134}$$

As ωt changes from zero to $\pi/2$, the direction of the vector **E** changes from $+x$ to $-y$. This can only happen if the rotation of the vector **E** is clockwise (see **Figure** 12.19c). From the right-hand rule, this is then a left elliptically polarized wave.

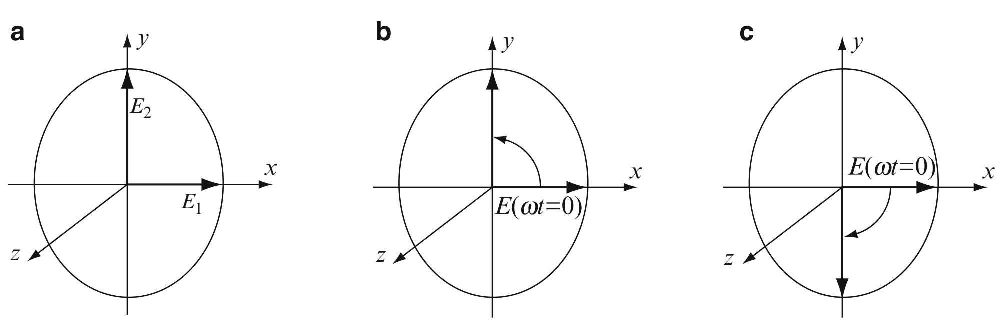

Figure 12.19 (**a**) An elliptically polarized wave. (**b**) The electric field rotates counterclockwise: the wave is right elliptically polarized. (**c**) The electric field rotates clockwise: the wave is left elliptically polarized. In circular polarization, $E_1 = E_2$

Case 3: $E_1 = E_2$ and $\varphi = -\pi/2$. This is clearly similar to case 1 above. If we set $E_1 = E_2 = E_0$ in **Eq. (12.131)**, we get

$$E_x^2 + E_y^2 + E_0^2 \tag{12.135}$$

This is the equation of a circle; therefore, the polarization is circular. Since **Eq. (12.129)** remains unchanged, except for the fact that $E_1 = E_2$, the wave is right circularly polarized. (see **Figure 12.19b** but with $E_1 = E_2$).
Clearly then, circular polarization is a special case of elliptical polarization.

Case 4: $E_1 = E_2$ and $\varphi = +\pi/2$. From the discussion in **case 3** and **case 2**, this is a left circularly polarized wave (see **Figure 12.19c** but with $E_1 = E_2$).

It is also worth noting the following:

(1) If $\varphi = 0$, the wave in **Eq. (12.127)** becomes

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_1 \cos(\omega t - \beta z) + \hat{\mathbf{y}} E_2 \cos(\omega t - \beta z) \tag{12.136}$$

This is identical in form to **Eq. (12.126)** and the wave is, therefore, a ***linearly polarized*** wave with polarization at an angle $\tan^{-1}(E_2/E_1)$ with respect to the x component.

(2) If $\varphi = 180°$, the wave in **Eq. (12.127)** is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_1 \cos(\omega t - \beta z) + \hat{\mathbf{y}} E_2 \cos(\omega t - \beta z + \pi) = \hat{\mathbf{x}} E_1 \cos(\omega t - \beta z) - \hat{\mathbf{y}} E_2 \cos(\omega t - \beta z) \quad [\mathrm{V/m}] \tag{12.137}$$

This is again a linearly polarized wave at an angle $\tan^{-1}(-E_2/E_1)$ with respect to the x component.

(3) From (2) and (1), it is clear that linear polarization is also a special case of the general elliptical polarization.

(4) From **Eqs. (12.127)** and **(12.131)**, we can also conclude that an elliptically polarized wave can always be written as the superposition of two linearly polarized waves. Two superposed linearly polarized waves produce an elliptically polarized wave if the amplitudes of the two waves are different and if there is a phase difference between the two. The phase difference also defines the sense of rotation of the wave. If the phase difference between the two linearly polarized waves is zero, the result is a linearly polarized wave as a special case of elliptical polarization. If the two linearly polarized waves have equal amplitudes but there is also a phase difference between the two, the superposed wave is a circularly polarized wave.

The lead or lag in **Eqs. (12.127)** and **(12.128)** were set to 90° because the results lead to simple expressions. However, the phase difference between two waves can be arbitrary. If this is the case, the waves are elliptically polarized (or circularly polarized if the amplitudes are the same). The only difference is that the ellipse the vectors **E** describe is rotated in space (its axes do not coincide with the x and y axes).

Example 12.19 The following waves are given:

(a) $\mathbf{E}(z,t) = -\hat{\mathbf{y}}\, 25 e^{-0.001z} \cos\left(10^3 t - 1000z\right) \quad [\mathrm{V/m}]$.

(b) $\mathbf{H}(z) = -\hat{\mathbf{x}} H_0 e^{-j\beta z} + \hat{\mathbf{y}}\, 2H_0 e^{-j\beta z} \quad [\mathrm{A/m}]$.

(c) $\mathbf{H}(z) = -\hat{\mathbf{y}} H_0 e^{-j\beta z} + j\hat{\mathbf{x}} H_1 e^{-j\beta z} \quad [\mathrm{A/m}]$.

Find the polarization in each case.

Solution: The polarization is obtained by systematic application of the four steps in **Section 12.8.2**. However, in **(b)** and **(c)**, the electric field intensities must be found first.

(a) The electric field intensity is always directed in the y direction. As the wave propagates, its amplitude is attenuated and its phase changes. Thus, the wave is linearly polarized in the y direction. Another way to look at it is to formally apply the four-step method given in **Section 12.8.2**.

Step a is not necessary because the field is given in the time domain.

Step b: We set $z = 0$:

$$\mathbf{E}(z = 0, t) = -\hat{\mathbf{y}}\, 25\cos\left(10^3 t\right) \quad [\mathrm{V/m}]$$

Step c: As time changes, the vector **E** may be either in the positive y or negative y direction. The field is linearly polarized in the y direction.

Step d: For a linearly polarized wave, there can be no rotation. This can be seen from the fact that for any two values of t the vector remains on the y axis.

(b) The magnetic field intensity has two components: one in the y direction and one in the x direction with amplitude half that of the y component. The two components are in phase; therefore, the polarization is linear, but for proper characterization, we must first find the electric field intensity. This is found from Ampere's law:

$$\nabla \times \mathbf{H} = \hat{\mathbf{x}}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\varepsilon\mathbf{E}$$

With $H_z = 0$, $\partial H_y/\partial x = 0$, and $\partial H_x/\partial y = 0$ and calculating the derivatives $\partial H_x/\partial z$ and $\partial H_y/\partial z$, we get

$$j\omega\varepsilon\mathbf{E} = \hat{\mathbf{x}}\, j\beta 2H_0 e^{-j\beta z} + \hat{\mathbf{y}}\, j\beta H_0 e^{-j\beta z}$$

Dividing both sides by $j\omega\varepsilon$ and setting $\eta = \beta/\omega\varepsilon$ gives the expression for the electric field intensity:

$$\mathbf{E} = \hat{\mathbf{x}}\,\eta 2H_0 e^{-j\beta z} + \hat{\mathbf{y}}\,\eta H_0 e^{-j\beta z} \quad [\mathrm{V/m}]$$

Now, we apply steps **a** through **d** to find the polarization and sense of rotation (if any).
Step a: The vector **E** is written in the time domain:

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{\hat{\mathbf{x}}\,\eta 2H_0 e^{-j\beta z}e^{-j\beta t} + \hat{\mathbf{y}}\,\eta H_0 e^{-j\beta t}\right\} = \hat{\mathbf{x}}\,\eta 2H_0\cos(\omega t - \beta z) + \hat{\mathbf{y}}\,\eta H_0\cos(\omega t - \beta z) \quad [\mathrm{V/m}]$$

Step b: We set $z = 0$:

$$\mathbf{E}(z=0,t) = \hat{\mathbf{x}}\,\eta 2H_0\cos(\omega t) + \hat{\mathbf{y}}\,\eta H_0\cos(\omega t) \quad [\mathrm{V/m}]$$

Step c: At $t = 0$, the vector **E** has components $2\eta H_0$ in the positive x direction and ηH_0 in the positive y direction. This ratio remains constant as t changes. Thus, **E** is linearly polarized at an angle equal to $\tan^{-1}(Hy/Hx) = \tan^{-1}(1/2) = 26°34'$ with respect to the positive x axis (see **Figure 12.20a**).

(c) First, we find the electric field intensity. Applying Ampere's law as in **(b)**, setting $H_z = 0$, $\partial H_y/\partial x = \partial H_x/\partial y = 0$, and calculating the derivatives $\partial H_x/\partial z$ and $\partial H_y/\partial z$ gives

$$j\omega\varepsilon\mathbf{E} = -\hat{\mathbf{x}}\, j\beta H_0 e^{-j\beta z} + \hat{\mathbf{y}}\, j\beta H_1 \mathrm{e}^{-j\beta z}$$

Now, we divide by $j\omega\varepsilon$ and set $\eta = \beta/\omega\varepsilon$:

$$\mathbf{E} = -\hat{\mathbf{x}}\,\eta H_0 e^{-j\beta z} - \hat{\mathbf{y}}\, j\eta H_1 e^{-j\beta z} \quad [\mathrm{V/m}]$$

Step a: The electric field intensity in the time domain is

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{-\hat{\mathbf{x}}\,\eta H_0 e^{-j\beta z}e^{-j\omega t} - \hat{\mathbf{y}}\, j\eta H_1 e^{-j\beta z}e^{-j\omega t}\right\} = -\hat{\mathbf{x}}\,\eta H_0\cos(\omega t - \beta z) - \hat{\mathbf{y}}\,\eta H_1\cos(\omega t - \beta z + \pi/2) \quad [\mathrm{V/m}]$$

where $j = e^{j\pi/2}$ was used.
Step b: Setting $z = 0$ gives

$$\mathbf{E}(z=0,t) = -\hat{\mathbf{x}}\,\eta H_0\cos(\omega t) - \hat{\mathbf{y}}\,\eta H_1\cos(\omega t + \pi/2) = -\hat{\mathbf{x}}\,\eta H_0\cos(\omega t) - \hat{\mathbf{y}}\,\eta H_1\sin(\omega t) \quad [\mathrm{V/m}]$$

This is clearly an elliptically polarized wave [see **Eqs. (12.129)** through **(12.131)**] since $H_0 \neq H_1$ and, as t changes, the vector **E** describes an ellipse (**step c**).
Step d: The rotation of **E** is found by setting $\omega t = 0$ and $\omega t = \pi/2$. These give

$$\mathbf{E}(z=0,\omega t=0) = -\hat{\mathbf{x}}\,\eta H_0, \quad \mathbf{E}(z=0,\omega t=\pi/2) = -\hat{\mathbf{y}}\,\eta H_1 \quad [\mathrm{V/m}]$$

This indicates rotation in the counterclockwise direction. The wave is ***right elliptically polarized*** (see **Figure 12.20b**).

Note: Because polarization is defined on the electric field intensity, we had to first find the electric field intensity from the magnetic field intensity in **(b)** or **(c)**. It is also possible to find the polarization of the magnetic field intensity since the magnetic field intensity in a plane wave is always perpendicular to the electric field intensity and rotates with the electric field intensity (if there is any rotation). See **Exercise 12.10**.

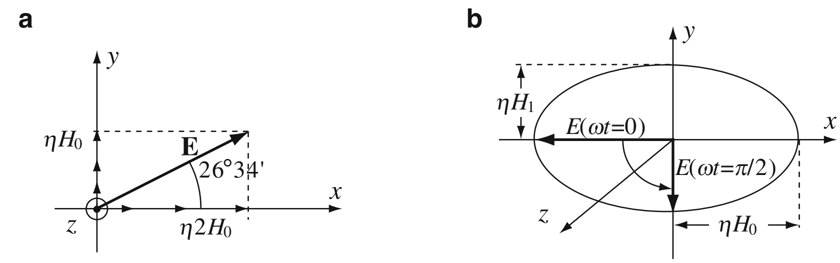

Figure 12.20 Identification of polarization. (**a**) The wave in **Example 12.19b** shown in the second quadrant. (**b**) The wave in **Example 12.19c** and its sense of rotation (assuming $H_0 > H_1$)

Exercise 12.10 In **Example 12.19**, find the polarization of the magnetic field intensity.

Answer **(a) H** is linearly polarized in the x direction. **(b) H** is linearly polarized at $116°34'$ with respect to the positive x axis. **(c) H** is right elliptically polarized.

12.9 Applications

Application: Communication with Spacecraft One of the most challenging communication problems is that with distant spacecraft. Although communication with objects in space is relatively simple and in many ways seems to be easier than communication on Earth, it has its own challenges. One of these is the vast distances involved. When communicating from Earth stations to spacecraft, large power and special antennas can be used, but when communicating from spacecraft to Earth, both power and antenna size are very limited. As spacecraft push further in space, the available power decreases because of reduced solar intensity, yet the requirements for range (and therefore for power) increase. A uniquely interesting example is offered by the Voyager spacecraft. Voyager 2 was launched on August 20, 1977, followed by Voyager 1 on September 15, 1977. In February 2014, Voyager 1 was at a distance of 19 billion km from Earth traveling in interstellar space, which it entered in August 2012. It is expected to cease transmission by 2025 when its RTG (radioisotope thermoelectric generator) output will not be sufficient to power the spacecraft (The RTG uses Plutonium 238 which has a half lifetime of 87.74 years). Communication with the spacecraft at such distances takes over 17 h, 8 m, each way. More interesting is the fact that the spacecraft transmitter is a mere 23 W (at launch), perhaps much less because of deterioration of its plutonium-powered power sources after some 37 years of continuous operation (compare this small power with the 50 kW of some AM stations or the 50 MW some radar equipment use). Or compare that to some portable radios (such as citizens band (CB) radios) that use between 3 and 5 W of power for a range of a few kilometers. From this vast distance the waves travel, it is also clear that the attenuation in space is rather small but is not zero. Both the time delay and the attenuation of waves will be a big problem in any long-range mission to the stars, when undertaken. In fact, because of these limitations, any deep-space exploration will have to be autonomous, with the spacecraft traveling, perhaps for generations, and returning with information that we will never get but future generations will. It may sound discouraging, but it really is quite exciting: it is like planting a tree that will most certainly outlast us. Each of the Voyager spacecraft carries a golden record with recordings and images representing life on earth. Should it encounter any civilization in the outer space, they should be able to identify the source of the spacecraft. See also the introduction to **Chapter 11** for a short description of the Pioneer 10 spacecraft.

Application: Range of TV and Radio Transmission TV and radio stations are regulated as to frequency, maximum power allowable, type of polarization, and other aspects of their operation. However, the range of a station depends, among other things, on attenuation in air. This, in turn, depends on a host of environmental conditions, including amount of moisture in air, pressure, pollutants, surface conditions, and others. Because of all these, the range, particularly of TV stations, is rather short. Also, some types of transmissions, such as microwaves, travel in a line-of-sight manner. Any obstruction such as hills, buildings, etc., prevents reception at the obstructed site. The range of a transmission system is a rather complex problem which must take into account antennas, attenuation, environmental conditions, and background noise, among others. Because of attenuation, the range of a TV station transmitting 50 kW of power is no more than about 100 km and even this range may be too large for good reception. AM transmission occurs at much lower frequencies (540 kHz–1.6 MHz). At these frequencies the attenuation is generally lower and the range is longer. On the other hand, FM transmission is in the VHF range (88–106 MHz) and therefore has a range similar to that of TV transmission. Typically, the attenuation constant in air (below about 3 GHz) is about 0.01 dB/km. Although we cannot calculate the range of transmission accurately, we can get a pretty good idea for the range using known attenuation constants and assuming an isotropic antenna (antenna which transmits uniformly in all directions). This then gives a worst-case maximum range which can be improved through use of more directional transmission, better antennas, etc.

Application: Superconducting Power Transmission Hollow conductors for power transmission at superconducting temperatures. The idea of skin depth[5] has been proposed for an unusual application: transmission of power in superconducting cables. Assuming that superconductivity will always require refrigeration, one proposed system is to use hollow conductors in which the refrigerant is passed keeping the temperature of the cables low enough for superconductivity to be maintained. Because at superconducting temperatures the conductivity is very high and the depth at which current exists is small, only the outer surface and a small depth below it will conduct current. A proposed method is to immerse the conductor in the form of a thin film of superconducting material coated on a metallic tube, in liquid helium or, when high-temperature superconductors are available, in liquid nitrogen. A possible method of using superconductors is demonstrated by the simplified superconducting power conductor in **Figure** 12.21 and could carry vast amounts of power with little or no losses. This is particularly attractive for power distributions where losses in conductors account for some 3–5 % of all losses in power generation and distribution.

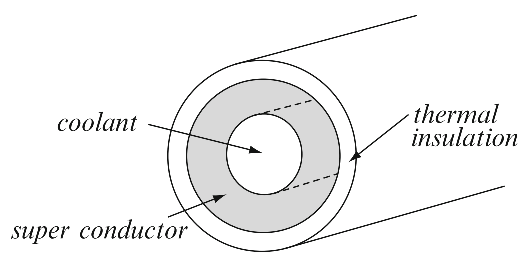

Figure 12.21 Structure of a superconducting cable

Application: Optical Fiber Magnetometer The use of optical fibers for communication is well established, but optical fibers have many other applications. One useful application is in the measurement of very low magnetic fields. The principle is quite simple and is based on two fundamental properties. One is the magnetostrictive properties of some materials and the second is the change in phase of a wave as the length of path it travels changes. The method is shown in **Figure** 12.22. Two very long optical fibers of identical length are connected to the same laser source. One fiber is coated with a magnetostrictive material. When no magnetic fields are present, the paths of light are identical and the output of the two fibers is in phase. Phase comparison between the two fibers shows a zero output. If both fibers are placed in a magnetic field, the magnetostrictive fiber changes its length by contraction. The contraction is rather small, and the total change in length is given as

[5] Skin depth per se does not apply to superconducting materials. There is, however, an equivalent relation that governs depth of current penetration in superconductors usually written as $\delta = \sqrt{\Lambda/4\pi}$, where Λ is known as the London order parameter. This relation is called the London relation after Heintz London (1907–1970) and his older brother Fritz London (1900–1954), who, among other important contributions to superconductivity, studied AC losses in superconductors.

$$\frac{\Delta l}{l} = cB_0^2$$

where c is the magnetostrictive material constant given in $1/\mathrm{T}^2$. $c = 10^{-4}$ $[1/\mathrm{T}^2]$ for nickel. Even though the effect is rather small, a very small change in length of the fiber will change the phase considerably. Since the propagation is at optical frequencies, the wavelength is of the order of a few hundred nanometers. For example, if a He–Ne (helium–neon) laser emitting at 633 nm (red) is used, a change in length of 100 nm changes the phase by $200\pi/633 = 0.31\pi$ or $28°$. The length of the fibers, l, can be as long as we wish since the fibers can be placed on spools or in any convenient configuration. Sensitivities below 10^{-9} T are obtainable with a device which is a relatively simple, rugged, passive device.

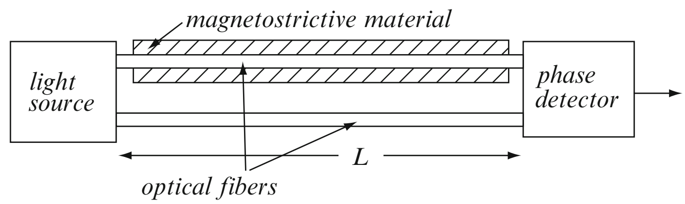

Figure 12.22 The principle of a magnetostrictive optical fiber magnetometer

12.10 Experiments

Experiment 1 (Shielding of Transmissions. Demonstrates: Unintended Shielding of High-Frequency Waves). The phenomenon of shielding of transmission is well known. One way to experiment with it is to listen to your car radio as you enter or exit from an underground parking lot. Observe how the reception changes as you enter or exit. Much of the shielding is due to iron bars used for the reinforcement of concrete. The concrete itself as well as soil are somewhat conducting, contributing to the effect. Therefore, if you were to try to receive transmissions deep into a mine, you may not be able to, depending on the local conditions (soil, rock, mineral properties, depth, moisture content, and the like). This places considerable restrictions on wireless communication in underground mines.

Experiment 2 (Shielding in Metallic Structures. Demonstrates: Design of Shielding Structures)

(a) Place a small, battery-operated radio in a metal box. The radio will cease to receive. Any metal box will do, including perforated boxes. Try the same with a metallic birdcage. Make sure the antenna does not touch the metallic structure. Tune to a weak radio station and try both FM and AM stations. In most cases, the reception should either cease or be reduced significantly. What happens when you touch the metal structure with the antenna?

(b) A car radio will not operate at all without an antenna, but a small portable radio will. The reason is simple. Car radios are constructed in a metal box to shield them from engine and other electronic noise.

Experiment 3 (Polarization. Demonstrates: Polarization of Waves and Its Relation to Antennas). Use a TV receiver or a radio and tune it to a relatively distant station. Rotate the antenna for maximum reception. Show that polarization is either vertical or horizontal, or both, depending on the device: radio (AM or FM) or TV.

Experiment 4 (Propagation in Water. Demonstrates: Propagation and Attenuation in Lossy Dielectrics). Take a battery-operated radio and seal it in a plastic bag. Operate the radio so that it receives your favorite station. Immerse the radio in water (pool or nonconductive bathtub). What happens with an FM and an AM station? To listen to the radio, you can go with it into the water or insert a tube into the bag and listen through the tube. Everything must be sealed properly and you must use a battery-operated radio to avoid shock and also because many small radios use the cord as an antenna. Also, do not use any of the radios that require an earphone to operate: these use the earphone cable as an antenna. When you go swimming, take the radio with you. Check the reception underwater at different depths. Preferably, use a weak station; otherwise there may be enough signal strength to penetrate into the water. If you can, repeat the experiment in seawater. Note the attenuation of the signal as the station disappears.

12.11 Summary

The fundamentals of wave propagation and the behavior of waves in various media are the subjects of the present chapter. We start with the source-free electromagnetic wave equation in general, lossy **[Eq. (12.16)]**, and lossless **[Eq. (12.18)]** media (see **Examples 12.3** and **12.4** and **Exercises 12.2** through **12.4**):

$$\nabla^2\mathbf{E} = j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E}) \tag{12.16}$$

and

$$\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0 \tag{12.18}$$

Wave equations identical in form may be written for **H**, **B**, **D**, **A**, or V and may also be written in the time domain.

Uniform plane waves are waves in which the amplitude and phase are constant at any point on any plane perpendicular to the direction of propagation of the wave. The form we assume is $\mathbf{E} = \hat{\mathbf{x}}E(z)$.

Solution of the lossless wave equation **[Eq. (12.18)]** for plane waves in lossless media is

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz} \quad [\text{V/m}] \tag{12.24}$$

$$k = \omega\sqrt{\mu\varepsilon} \quad [\text{rad/m}] \tag{12.23}$$

or, in the time domain,

$$E_x(z,t) = \text{Re}\{E_x(z)e^{j\omega t}\} = E_0^+\cos(\omega t - kz) + E_0^-\cos(\omega t + kz) \quad [\text{V/m}] \tag{12.25}$$

An arbitrary phase angle ϕ may also be added to either solution (due to, for example, the complex nature of E_0^+ or E_0^-). The first term is a forward-propagating wave (in the positive z direction), the second a backward-propagating wave (negative z direction in this case).

Properties of the Wave

$$\text{Phase velocity}: \quad v_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \left[\frac{\text{m}}{\text{s}}\right] \tag{12.28}$$

$$\text{In free space}: \quad v_p \approx 3\times 10^8 \quad [\text{m/s}] \tag{12.29}$$

$$\text{Wavelength}: \quad \lambda = \frac{v_p}{f} = \quad [\text{m}] \tag{12.30}$$

$$\text{Wave number}: \quad k = \frac{2\pi}{\lambda} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.31}$$

$$\text{Intrinsic impedance}: \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad [\Omega] \tag{12.37}$$

$$\text{In free space}: \quad \eta_0 \approx 377 \quad [\Omega] \tag{12.39}$$

Poynting Theorem, Poynting Vector, Power, and Power Density The ***Poynting vector*** gives the magnitude and direction of propagation of the instantaneous power density:

$$\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{H} \quad [\text{W/m}^2] \tag{12.53}$$

The ***Poynting theorem*** gives the net power entering or leaving a volume v, enclosed by area s:

$$\mathcal{P}(t) = \oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_v \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right) dv - \int_v \mathbf{E} \cdot \mathbf{J} dv \quad [\text{W}] \tag{12.52}$$

The first term on the right-hand side is the time rate of change of stored energy, the second is power due to source and induced currents. A net negative power (power flow into the volume) is called the ***receiver case***. Net positive power (out of the volume) is called the ***transmitter case***.

Time-averaged power density can be calculated from instantaneous power density or from the complex Poynting vector:

$$\mathcal{P}_{av} = \frac{1}{T}\int_0^T \mathcal{P}(t) dt \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{12.56}$$

or

$$\mathcal{P}_{av} = \frac{1}{2}\text{Re}\{\mathbf{E} \times \mathbf{H}^*\} \quad \left[\frac{\text{W}}{\text{m}^2}\right] \tag{12.59}$$

where $T = 1/f = 2\,\pi/\omega$ and * indicates the complex conjugate. The complex Poynting vector is $\mathcal{P}_c = \mathbf{E} \times \mathbf{H}^*$ [W/m^2]. The complex Poynting theorem may be written as

$$\oint_s (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = j\omega\int_v (\varepsilon\mathbf{E} \cdot \mathbf{E}^* - \mu\mathbf{H} \cdot \mathbf{H}^*) dv - \int_v \mathbf{E} \cdot \mathbf{J}_0^* dv - \int_v \sigma\mathbf{E} \cdot \mathbf{E}^* dv \quad [\text{W}] \tag{12.71}$$

where $\mathbf{E} \cdot \mathbf{J}_0^*$ may be negative or positive depending on the source of $\mathbf{J}_0^*$. The last term represents ohmic losses. Time-averaged power is usually calculated using **Eq. (12.71)**.

Time-averaged energy densities (electric and magnetic) are

$$w_{e(av)} = \frac{\varepsilon\mathbf{E} \cdot \mathbf{E}^*}{4}, \quad w_{m(av)} = \frac{\mu\mathbf{H} \cdot \mathbf{H}^*}{4} \tag{12.74}$$

Propagation of Plane Waves in General Media Given properties (ε,μ,σ) the wave equation is written in terms of the complex permittivity ε_c as

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon_c)\mathbf{E} \tag{12.81}$$

where

$$\varepsilon_c = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right] \tag{12.79}$$

The wave equation to solve is

$$\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = 0 \tag{12.84}$$

where

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \tag{12.83}$$

and

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \left[\frac{\text{Np}}{\text{m}}\right] \tag{12.95}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.96}$$

$\gamma = \alpha + j\beta$ is the ***propagation constant***, α is the ***attenuation constant***, and β the ***phase constant***. Phase velocity and wavelength are also dependent on conductivity **[**see **Eqs. (12.97)** and **(12.98)]**.

The ***intrinsic impedance*** is now complex:

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \quad [\Omega] \tag{12.102}$$

The main effect that is different than propagation in lossless media is attenuation of the waves. The solution for attenuated plane waves includes an attenuation factor:

$$E_x(z) = E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{\alpha z} e^{j\beta z} \quad [\text{V/m}] \tag{12.91}$$

or, in the time domain,

$$E_x(z,t) = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + E_0^- e^{\alpha z}\cos(\omega t + \beta z) \quad [\text{V/m}] \tag{12.93}$$

Low-Loss Dielectrics $\sigma/\omega\varepsilon \ll 1$. Approximations are defined as follows:

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad \left[\frac{\text{Np}}{\text{m}}\right] \tag{12.104}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}\left(1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right) \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.105}$$

$$\eta \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + \frac{j\sigma}{2\omega\varepsilon}\right) \quad [\Omega] \tag{12.109}$$

High-Loss Materials $\sigma/\omega\varepsilon \gg 1$. Approximations:

$$\alpha \approx \sqrt{\pi f\mu\sigma} \quad \left[\frac{\text{Np}}{\text{m}}\right], \quad \beta \approx \sqrt{\pi f\mu\sigma} \quad \left[\frac{\text{rad}}{\text{m}}\right] \tag{12.111}$$

$$\eta \approx (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta} = (1+j)\frac{\omega\mu\delta}{2} \quad [\Omega] \tag{12.116}$$

Skin depth is the depth at which the amplitude of the wave reduces to $1/e$ of its value:

$$\delta \approx \sqrt{\frac{1}{\pi f\mu\sigma}} = \frac{1}{\alpha} \quad [\text{m}] \tag{12.113}$$

Group velocity is the velocity of a packet of waves in a narrow range of frequencies. It is different from phase velocity except in perfect dielectrics

$$v_g = \frac{1}{d\beta/d\omega} \quad \left[\frac{\text{m}}{\text{s}}\right] \tag{12.122}$$

Dispersion is the frequency dependence of the phase velocity which causes waves of different frequencies to travel at different velocities. Perfect dielectrics are dispersionless.

Polarization of plane waves is the path described by the tip of the electric field intensity as it propagates in space toward the observer:

(1) Linear polarization—the tip of the electric field intensity describes a line.
(2) Circular polarization—the tip of the electric field intensity describes a circle.
(3) Elliptical polarization—the tip of the electric field intensity describes an ellipse.
(4) Rotation: circularly and elliptically polarized waves can rotate clockwise or counterclockwise as they propagate. Counterclockwise rotation is said to be ***right elliptically*** (or ***circularly) polarized*** because it follows the right-hand rule—the thumb is in the direction of propagation of the wave and the curled fingers show the direction of rotation of the electric field intensity. ***Left circularly*** (or ***elliptically) polarized*** waves rotate clockwise as they propagate toward the observer.

Problems

The Time-Dependent Wave Equation

12.1 The Wave Equation. Starting with the general time-dependent Maxwell's equations in a linear, isotropic, homogeneous medium, write a wave equation in terms of the electric field intensity.

(a) Show that if you neglect displacement currents, the equation is not a wave equation.
(b) Write the source-free wave equation from the general equation you obtained.

12.2 Source-Free Wave Equation. Obtain the source-free time-dependent wave equation for the magnetic flux density in a linear, isotropic, homogeneous medium.

The Time-Harmonic Wave Equation

12.3 Time-Harmonic Wave Equation. Using the source-free Maxwell's equations, show that a Helmholtz equation can be obtained in terms of the magnetic vector potential. Use the definition $\mathbf{B} = \nabla \times \mathbf{A}$ and a simple medium (linear, isotropic, homogeneous material). Justify the choice of the divergence of $\mathbf{A}$.

12.4 The Helmholtz Equation for D. Using Maxwell's equations, find the Helmholtz equation for the electric flux density $\mathbf{D}$ in a linear, isotropic, homogeneous material.

12.5 The Helmholtz Equation for B. Find the Helmholtz equation in terms of the magnetic flux density using the source-free time-harmonic Maxwell's equations in a linear, isotropic, homogeneous material.

12.6 The Electric Hertz Potential. In a linear, isotropic, homogeneous medium, in the absence of sources, the Hertz vector potential $\mathbf{\Pi}_e$ may be defined such that $\mathbf{H} = j\omega\varepsilon\nabla \times \mathbf{\Pi}_e$:

(a) Express the electric field intensity in terms of $\mathbf{\Pi}_e$.
(b) Show that the Hertz potential satisfies a homogeneous Helmholtz equation provided a correct gauge is chosen. What is this appropriate gauge?

Solution for Plane Waves

12.7 Plane Wave. A plane wave propagates in the positive x direction in free space. The wave is given at $x = 0$ as $\mathbf{E}(0) = \hat{\mathbf{z}}\, 5\cos\left(10^9 \pi t\right)$ [V/m]. If properties of free space are $\varepsilon = \varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m], find $\mathbf{E}(x = 20)$.

12.8 Plane Wave. A wave propagates in free space and its electric field intensity is $\mathbf{E} = \hat{\mathbf{x}}\, 100e^{-j220z} + \hat{\mathbf{x}}\, 100e^{-j220z}$ [V/m]:

(a) Show that $\mathbf{E}$ satisfies the source-free wave equation.
(b) What are the wave's phase velocity and frequency?

The Poynting Vector

12.9 A simple and common use of the Poynting vector is identification of direction of propagation of a wave or the direction of fields in space. Consider the magnetic field intensity of a plane electromagnetic wave propagating in free space:

$$\mathbf{H} = \hat{\mathbf{x}} H_0 e^{j\beta y} + \hat{\mathbf{z}} H_1 e^{j\beta y} \quad [\mathrm{A/m}]$$

(a) Calculate the electric field intensity using the properties of the Poynting vector.
(b) Show that the result in **(a)** is correct by substituting the magnetic field intensity into Maxwell's equations and evaluating the electric field intensity through Maxwell's equations.

12.10 Application: **Power Relations in a Microwave Oven**. The peak electric field intensity at the bottom of a microwave oven is equal to 2500 V/m. Assuming that this is uniform over the area of the oven which is equal to 400 cm^2, calculate the peak power the oven can deliver. Permeability and permittivity are those of free space.

12.11 Application: **Heating Food in a Microwave Oven**. A frozen pizza is marketed to be heated in a microwave oven. As an engineer you are asked to write heating instructions, specifically how long should it be heated to reach a proper temperature. The average residential microwave oven is 50 cm wide, 40 cm deep, and 30 cm high. The oven has a low and a high heating level. At low, it produces a time averaged power of 500 W, whereas at high, the power is 1000 W. We will assume the pizza is placed flat on the bottom of the oven and any power coupled into the pizza enters from above. The pizza is 25 cm in diameter, 1.5 cm thick, and is 75 % water by volume. The microwave oven heats the water in the pizza. The frozen pizza is at –20 °C and must be heated to 75 °C. Heat capacity of water is 4.1885 J/(g·K) and the latent heat (of melting ice) is 334 J/g (heat capacity is the energy needed to raise the temperature of one gram of substance (water in this case) by 1 degree Kelvin, and latent heat is the energy required to melt a gram of ice at 0 °C to water at 0 °C). Assume that the heat transfer from the electromagnetic waves to the pizza is 80 % efficient and calculate:

(a) The time it takes to heat the pizza on the low setting of the oven.
(b) The time it takes to heat the pizza on the high setting of the oven.
(c) The cost in electricity to heat the pizza if a kW·h costs $0.16.

12.12 Power Dissipation in Cylindrical Conductor. A cylindrical conductor of radius R and infinite length carries a current of amplitude I [A] and frequency f [Hz]. The conductivity of the conductor is σ [S/m] (**Figure 12.23**). Calculate the time-averaged dissipated power per unit length in the conductor, neglecting displacement currents in the conductor. Assume the current is uniformly distributed throughout the cross section.

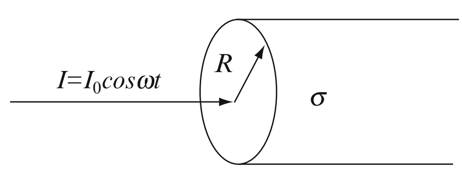

Figure 12.23

12.13 Application: **Power Radiated by an Antenna**. An antenna produces an electric field intensity in free space as follows:

$$\mathbf{E} = \hat{\boldsymbol{\theta}} \frac{12\pi}{R} e^{-j2\pi R} \sin\theta \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

where R is the radial distance from the antenna and the field is described in a spherical coordinate system. The field of the antenna behaves as a plane wave in the spherical system of coordinates. Calculate:

(a) The magnetic field intensity of the antenna.
(b) The time-averaged power density at a distance R from the antenna.
(c) The total radiated power of the antenna.

12.14 Application: **Electromagnetic Radiation Safety**. The allowable time-averaged microwave power density exposure in industry in the United States is 10 $\mathrm{mW/cm^2}$. As a means of understanding the thermal effects of this radiation level (nonthermal effects are not as well defined and are still being debated), it is useful to compare this radiation level with thermal radiation from the Sun. The Sun's radiation on Earth is about 1,400 W/m^2 (time averaged). To compare the fields associated with the two types of radiation, view these two power densities as the result of a Poynting vector. Calculate:

(a) The electric and magnetic field intensity due to the Sun's radiation on Earth.
(b) The maximum electric and magnetic field intensities allowed by the standard. Compare with that due to the Sun's radiation.

12.15 Stored Energy. In a region of space where there are no currents, the time-averaged pointing vector equals 120 W/m^2. Assume that this power density is uniform on the surface of a sphere of radius $a = 0.1$ m, pointing outwards. Calculate the total stored energy in the sphere. The frequency is 1 GHz and the sphere has properties of free space.

Propagation in Lossless, Low-Loss, and Lossy Dielectrics

12.16 Energy Density in Dielectrics. A plane wave of given frequency propagates in a perfect dielectric:

(a) Calculate the time-averaged stored electric energy density and show that it is equal to the magnetic volume energy density.
(b) Suppose now the dielectric is lossy: calculate the time-averaged-stored electric and magnetic energy densities. Are they still the same?

12.17 Propagation in Lossy Media. Seawater has a conductivity of 4 S/m. Its permittivity depends on frequency; relative permittivity at 100 Hz is 80, at 100 MHz it is 32, and at 10 GHz it is 24:

(a) How can seawater be characterized in terms of its loss at these three frequencies? That is, is seawater a low-loss, high-loss, or a general lossy medium for which no approximations can be made?
(b) Calculate the intrinsic impedance at the three frequencies.
(c) What can you conclude from these calculations for the propagation properties of seawater?

12.18 Generation of a Plane Wave in a Lossy Dielectric. A very thin conducting layer carries a surface current density J_0 [A/m] as shown in **Figure 12.24**. The frequency is f and the current is directed in the positive z direction. The layer is immersed in seawater, which has permittivity ε, permeability μ_0, and conductivity σ. If the layer is at $x = 0$, calculate the electric and magnetic fields at $x = x_0$. Given: $J_0 = 1$ A/m, $f = 100$ MHz, $\sigma = 4$ S/m, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\varepsilon_r = 80$, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $x_0 = 1$ m.

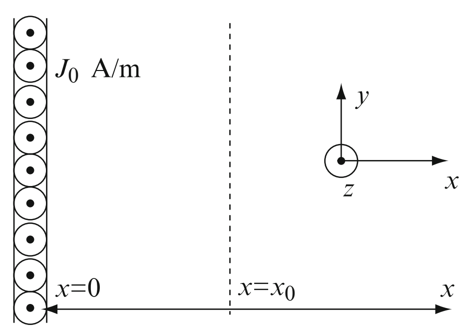

Figure 12.24

12.19 Application: **Communication in the Atmosphere**. A parabolic antenna of radius b [m] transmits at a frequency f to a receiving antenna a distance d [m] away. The receiving antenna is also of radius b [m] and the transmission is parallel to the ground, in the atmosphere, as in **Figure 12.25a**. Assume that the wave propagates as a plane wave and the beam remains constant in diameter (same diameter as the antennas). Use the following values: $b = 1$ m, $d = 200$ km, $\varepsilon = \varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m] $\sigma = 2 \times 10^{-7}$ S/m, and $f = 300$ MHz.

(a) Calculate the time-averaged power the transmitting antenna must supply if the receiving antenna must receive a magnetic field intensity of magnitude 1 mA/m.
(b) In an attempt to reduce the power required, the transmission is directed to a satellite which contains a perfect reflector, as shown in **Figure 12.25b**. The waves propagate through the atmosphere, into free space to the satellite and back to the receiving antenna. If the satellite is at a height h [m] and the waves propagate in the atmosphere

for a distance q [m] in each direction, what is the power needed in this case for the same reception condition as in **(a)**? In addition to the values given in **Figure 12.25a**, $h = 36{,}000$ km, $q = 20$ km, and $\sigma = 0$ in free space. Compare the result with that in **(a)**.

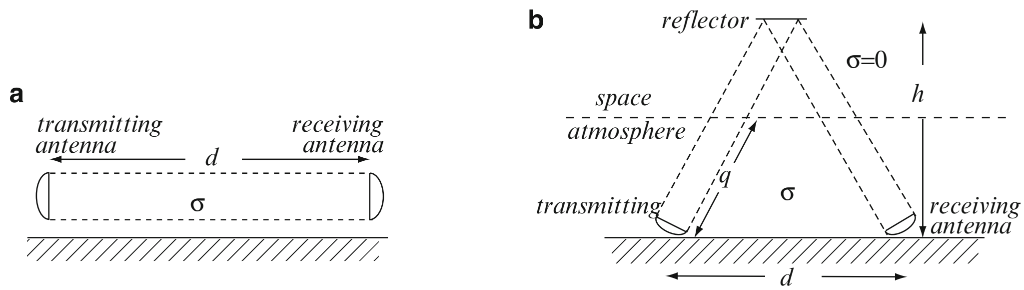

Figure 12.25

12.20 **Power Relations in the Atmosphere**. A plane wave propagates in the atmosphere. Properties of the atmosphere are ε_0 [F/m], μ_0 [H/m], and σ [S/m]. The electric field intensity has an amplitude E_0 [V/m] in the x direction and E_0 [V/m] in the z direction. The frequency of the wave is f [Hz]. Given: $E_0 = 100$ V/m, $\sigma = 10^{-6}$ S/m, $f = 100$ MHz. Find:

(a) The direction of propagation.
(b) The instantaneous power per unit area in the direction of propagation.
(c) The magnitude of the magnetic field intensity after the wave has propagated a distance $d = 10$ km.

12.21 **Application**: **Radar Detection and Ranging of Aircraft**. A radar antenna transmits 50 kW at 10 GHz. Assume transmission is in a narrow beam, 1 m^2 in area, and that within the beam, waves are plane waves. The wave is reflected from an aircraft but only 1 % of the power propagates in the direction of the antenna. If the airplane is at a distance of 100 km, calculate the total power received by the antenna. Assume permittivity and permeability of free space and conductivity of 10^{-7} S/m.

12.22 **Application**: **Fiber Optics Communication**. Two optical fibers are used for communication. One is made of glass with properties $\mu = \mu_0$ [H/m] and $\varepsilon = 1.75\varepsilon_0$ [F/m] and has an attenuation of 2 dB/km. The second is made of plastic with properties $\mu = \mu_0$ and $\varepsilon = 2.5\varepsilon_0$ and attenuation of 10 dB/km. Suppose both are used to transmit signals over a length of 10 km. The input to each fiber is a laser, operating at a free-space wavelength of 800 nm and input of 0.1 W. Calculate:

(a) The power available at the end of each fiber.
(b) The wavelength, intrinsic (wave) impedance, and phase velocity in each fiber.
(c) The phase difference between the two fibers at their ends.

12.23 **Application**: **Wave Properties and Remote Sensing in the Atmosphere**. A plane wave of frequency f propagates in free space and encounters a large volume of heavy rain. The permittivity of air increases by 5 % due to the rain:

(a) Calculate the intrinsic impedance, phase constant, and phase velocity in rain and the percentage change in wavelength.
(b) Compare the properties calculated in **(a)** with those in free space. Can any or all of these be used to monitor atmospheric conditions (such as weather prediction)? Explain.

12.24 **Application**: **Attenuation in the Atmosphere**. Measurements with satellites show that the average solar radiation (solar constant) in space is approximately 1400 W/m^2. The total radiation reaching the surface of the Earth on a summer day is approximately 1100 W/m^2. 50 % of this radiation is in the visible range. To get some insight into the radiation process, assume an atmosphere which is 100 km thick. From this calculate the average attenuation constant in the atmosphere over the visible range assuming it is constant throughout the range and that attenuation is the only loss process.

Propagation in High-Loss Dielectrics and Conductors

12.25 Intrinsic Impedance in Copper. Copper at 100 MHz has the following properties: $\mu = \mu_0$ [H/m], $\varepsilon = \varepsilon_0$ [F/m], and $\sigma = 5.7 \times 10^7$ S/m. Calculate the intrinsic impedance of copper:

(a) Using the exact formula for general lossy materials.
(b) Assuming a high-loss material. Compare with **(a)** and with the intrinsic impedance of free space.

12.26 Propagation of Microwaves in Metals. A microwave oven operates at 2450 MHz and at a time-averaged power of 1000 W. The microwave beam, which can be assumed to be a plane wave, is incident on a copper foil, 1 m^2 in area and 10 μm thick. Assuming that the electric field is parallel to the foil and that only 2 % of the incident electric field intensity enters the foil, calculate the amplitude of the electric field intensity just below the other surface of the foil (but still in the copper foil). Use ε_0 [F/m], μ_0 [H/m], and $\sigma = 0$ for free space and ε_0 [F/m], μ_0 [H/m], and $\sigma = 5.7 \times 10^7$ S/m for copper.

12.27 Skin Depth and Penetration in Lossy Media. Two plane waves propagate in two materials as shown in **Figure** 12.26:

(a) What is the ratio between the distances the waves travel in each material before the electric and magnetic field intensities are attenuated to 1% of their amplitude at the surface? Assume that the waves enter the materials without losses or reflections.
(b) What is the ratio between the phase velocities?

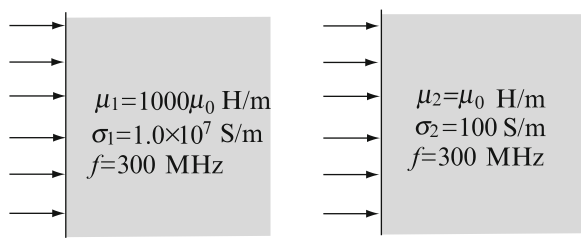

Figure 12.26

12.28 Measurement of Conductivity in Lossy Materials. In an attempt to measure conductivity of a material, a plane wave is applied to one surface of a slab and measured at the other surface. Suppose the electric field intensity just below the left surface of the material is measured as E_0 [V/m]. The electric field intensity at the right surface (again, just below the surface) is $0.1E_0$ [V/m]. Permeability and permittivity are those of free space, the material is known to have high conductivity, and the slab is $d = 10$ mm thick. The measurements are performed at 400 Hz. Calculate the conductivity of the material and its attenuation constant.

12.29 Application: **Underwater Communication**. Suppose a submarine could generate a plane wave and use it to communicate with another submarine in seawater. If the ratio between the amplitude at the receiver and that at the transmitter must be 10^{-12} or higher, what is the maximum range of communication at:

(a) 10 MHz.
(b) 100 Hz.

(Assume relative permittivity in both cases is 72 and conductivity of seawater is 4 S/m.)

12.30 Skin Depth in Conductors. Calculate the skin depth for the following conditions:

(a) Copper: $f = 10^{10}$ Hz, $\mu = \mu_0$ [H/m], $\sigma = 5.8 \times 10^7$ S/m.
(b) Mercury: $f = 10^{10}$ Hz, $\mu = \mu_0$ [H/m], $\sigma = 1 \times 10^6$ S/m.

12.31 Wave Impedance in Conductors. Calculate the intrinsic (wave) impedance of copper and iron at 60 Hz and 10 GHz. The conductivity of copper is 5.7×10^7 S/m and that of iron is 1×10^7 S/m. The permeability of copper is μ_0 [H/m] and that of iron is $1{,}000\mu_0$ [H/m] at 60 Hz and $10\mu_0$ [H/m] at 10 GHz:

(a) Compare these with the wave impedance in free space.
(b) What can you conclude from these calculations for the propagation properties of conductors in general and ferromagnetic conductors in particular?

12.32 Classification of Lossy Materials. In a material, $\varepsilon/\varepsilon_0 = 24$ and $\sigma = 4$ S/m. How do you classify this material for propagation purposes at 10 GHz? Explain.

12.33 Classification of Lossy Materials. A material has a conductivity of 0.01 S/m, permeability μ_0 [H/m], and relative permittivity of 72. Is this a conductor or a dielectric at 60 Hz and at 30 GHz?

12.34 Application: **Skin Depth and Communication in Seawater**. How deep does an electromagnetic wave transmitted by a radar operating at 3 GHz propagate in seawater before its amplitude is reduced to 10^{-6} of its amplitude just below the surface? Use the following properties: $\sigma = 4$ S/m, $\varepsilon = 24\varepsilon_0$ [F/m] (at 3 GHz), and $\mu = \mu_0$ [H/m]. How good is radar for detection of submarines? Explain.

12.35 Penetration of Light in Copper. Since light is an electromagnetic wave and electromagnetic waves penetrate in any material except perfect conductors, calculate the depth of penetration of light into a sheet of copper. The conductivity of copper is 5.7×10^7 S/m, and its permeability is that of free space. Assume the frequency of light (in mid-spectrum) is 5×10^{14} Hz.

12.36 Properties of Seawater at Different Frequencies. Seawater has relative permittivity of 81, permeability of free space, and conductivity of 4 S/m:

(a) Calculate the approximate range of frequencies over which seawater may be assumed to be a good conductor, assuming permittivity remains constant.
(b) Calculate the range of frequencies over which seawater may be assumed to be a good dielectric.

12.37 AC Current Distribution in a Conductor. A cable made of iron, with properties as shown in **Figure 12.27**, carries a current at 100 Hz. If the current density allowed (maximum) is 100 A/mm^2, find the current density at the center of the conductor ($\sigma = 1 \times 10^7$ S/m, $\mu = 20\,\mu_0$ [H/m], $r = 0.1$ m).

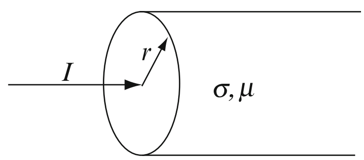

Figure 12.27

12.38 Application: **Skin Depth and Design of Cables for AC Power Distribution**. A thick wire is made of steel as shown in **Figure 12.28a**. If the maximum current density allowed at any point in the material is 10 A/mm^2:

(a) What is the total current the cable can carry at 60 Hz?
(b) To improve the current-carrying capability, a cable is made of 10 thinner wires (see **Figure 12.28b**) such that the total cross-sectional area is equal to that of the cable in **(a)**. What is the total current the new cable can carry for the same maximum current density, frequency, and material properties?
(c) Compare the results in **(a)** and **(b)** with the current the wire can carry at DC.

Note: Assume that the solution for plane waves applies here even though the surface is curved. This is, in general, applicable if the skin depth is small compared to the radius.

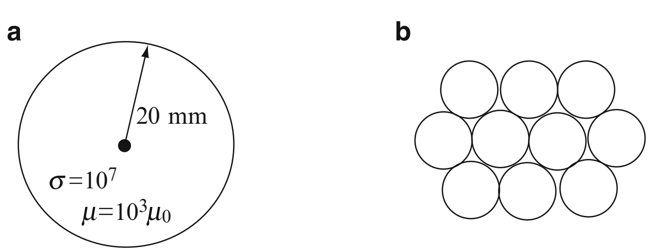

Figure 12.28

12.39 Application: **Electromagnetic Shielding**. A shielded room is designed so that high-frequency waves cannot penetrate into the room. The shield is made of a nonconducting, nonmagnetic material with a thin coating of conducting material on the outer surface. The shield must reduce the electric field intensity by a factor of 10^6 compared to the field outside at a frequency of 10 MHz. The conducting layer may be made of aluminum, copper, mu-metal, or a conducting polymer. Conductivities, permeabilities, cost, and mass of the three materials are given in the table below.

	Cu	Al	Mu-metal	Polymer
Conductivity [S/m]	5.7×10^7	3.6×10^7	0.5×10^7	0.001
Permeability [H/m]	μ_0	μ_0	$10^5\mu_0$	μ_0
Mass [kg/m^3]	8,960	2,700	7,800	1,200
Cost [unit/kg]	1	1.5	100	0.01

(a) Calculate the minimum thickness required for each of the four materials.
(b) Which material should we choose if the overall important parameter is: 1. Cost. 2. Mass. 3. Volume of conducting material. Use a unit area of the wall for comparison.

Dispersion and Group Velocity

12.40 Dispersion and Group Velocity. Under certain conditions, a wave propagates with the following phase constant:

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{\omega_c^2}{\omega^2}} \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

where ω [rad/s] is the angular frequency of the wave, and ω_c [rad/s] is a fixed angular frequency. Because this gives the relation between β and ω, it is a dispersion relation:

(a) Plot the dispersion relation for the wave: use $\omega_c = 10^7$, $10^7 \leq \omega \leq 2 \times 10^7$ [rad/s].
(b) What are the phase and group velocities?
(c) What happens at $\omega = \omega_c$? Explain.

12.41 Dispersion and Group Velocity. The dispersion relation for a wave is given as

$$\beta = \sqrt{\omega^2\mu\varepsilon - \frac{\pi^2}{a}} \quad \left[\frac{\text{rad}}{\text{m}}\right]$$

(a) Plot the dispersion relation. **a** is a constant.
(b) Find the group and phase velocities.

12.42 Phase and Group Velocities. Show that the following relation between group and phase velocity exists:

$$v_g = v_p + \beta\frac{dv_p}{d\beta} \quad \left[\frac{\text{m}}{\text{s}}\right]$$

12.43 Dispersion in Iron. Find and plot the relation between phase constant and angular frequency for a wave propagating in iron. Assume the relative permeability of iron is 1000 and its conductivity is 10^7 S/m:

(a) Plot the relation for all frequencies.
(b) Find the phase and group velocities at 100 Hz and at 100 MHz using the plot in **(a)**.

12.44 Phase and Group Velocities in Lossy Dielectrics. A plane wave with electric field intenstity $E = \hat{\mathbf{x}}\,100e^{-j2z}$[V/m] propagates in rubber, which has properties $\mu = \mu_0$ [H/m], $\varepsilon = 4\varepsilon_0$ [H/m], and $\sigma = 0.001$ S/m and may be considered to be a very low-loss dielectric. Calculate:

(a) The phase velocity in the material.
(b) The group velocity in the material.

(c) The energy transport velocity.
(d) What is the conclusion from the results in **(a)**–**(c)**?

12.45 Group Velocity and Dispersion in Low-Loss Media. A general low-loss medium is given in which the term $\sigma/\omega\varepsilon$ is small but not negligible:

(a) Calculate the group velocity.
(b) Plot the group velocity as a function of frequency.
(c) Is the medium dispersive and, if so, is the dispersion normal or anomalous? Explain.

Polarization of Plane Waves

12.46 Polarization of Plane Waves. The magnetic field intensity of a plane wave is given as $\mathbf{H}(x) = \hat{\mathbf{y}}\,10e^{-j\beta x}$ [A/m]. What is the polarization of this wave?

12.47 Polarization of Plane Waves. The magnetic field intensity of a plane wave is given as $\mathbf{H}(x) = \hat{\mathbf{y}}\,10e^{-j\beta x} + \hat{\mathbf{z}}\,10e^{-j\beta x}$ [A/m]. What is the polarization of this wave?

12.48 Polarization of Plane Waves. The electric field intensity of a wave is given as $\mathbf{E}(x) = \hat{\mathbf{y}}\,10\mathrm{e}^{-0.1x}\left(\mathrm{e}^{-j\beta x} + \mathrm{e}^{j\beta x}\right) + \hat{\mathbf{y}}\,10\mathrm{e}^{-0.1x}\left(\mathrm{e}^{-j\beta x} - \mathrm{e}^{j\beta x}\right)$ [V/m]. What is the polarization of this wave?

12.49 Polarization of Plane Waves. The electric field intensity of a wave is given as $\mathbf{E}(x,t) = \hat{\mathbf{y}}\,100\cos(\omega t - \beta x) + \hat{\mathbf{z}}\,200\cos(\omega t - \beta x - \pi/2)$ [V/m]. What is the polarization of this wave?

12.50 Polarization of Superposed Plane Waves. Two plane waves propagate in the same direction. Both waves are at the same frequency and have equal amplitudes. Wave A is polarized linearly in the x direction, and wave B is polarized in the direction of $\hat{\mathbf{x}} + \hat{\mathbf{y}}$. In addition, wave B lags behind wave A by a small angle θ. What is the polarization of the sum of the two waves?

12.51 Polarization of Plane Waves. The magnetic field intensity of a wave is given as $\mathbf{H}(x,t) = \hat{\mathbf{y}}\,100\cos(\omega t - \beta x) + \hat{\mathbf{z}}\,200\cos(\omega t - \beta x + \pi/2)$ [V/m]. What is the polarization of this wave?